

Existence of strong solutions to the steady Navier-Stokes equations for a compressible heat-conductive fluid with large forces

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Abstract

We prove that there exists a strong solution to the Dirichlet boundary value problem for the steady Navier-Stokes equations of a compressible heat-conductive fluid with large external forces in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), provided that the Mach number is appropriately small. At the same time, the low Mach number limit is rigorously verified. The basic idea in the proof is to split the equations into two parts, one of which is similar to the steady incompressible Navier-Stokes equations with large forces, while another part corresponds to the steady compressible heat-conductive Navier-Stokes equations with small forces. The existence is then established by dealing with these two parts separately, establishing uniform in the Mach number a priori estimates and exploiting the known results on the steady incompressible Navier-Stokes equations.

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1 Introduction

This paper is mainly concerned with the existence of strong solutions to the steady Navier-Stokes equations of a compressible heat-conductive fluid in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with large external forces:

$$\begin{cases} \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \varrho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} + \mathbf{g}, \\ c_v \varrho \mathbf{u} \cdot \nabla \Theta + p \operatorname{div} \mathbf{u} = \kappa \Delta \Theta + \Psi. \end{cases} \quad (1.1)$$

Here ϱ denotes the density, $\mathbf{u} \in \mathbb{R}^d$ the velocity, Θ the temperature, $p = R\varrho\Theta$ the pressure with $R > 0$ being the gas constant, $c_v > 0$ is the heat capacity at constant volume; \mathbf{f} is the density of external body force and \mathbf{g} is a given external force. The stress tensor \mathbb{S} and the dissipation function Ψ are defined by

$$\mathbb{S} = 2\mu D(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u} \mathbb{I}, \quad \Psi = 2\mu D(\mathbf{u}) : D(\mathbf{u}) + \lambda (\operatorname{div} \mathbf{u})^2 \geq 0,$$

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$D(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$ is the deformation tensor. The viscosity coefficients μ, λ satisfy $2\mu + d\lambda \geq 0$ and $\mu > 0, \kappa > 0$ is the heat conductivity coefficient. Moreover, the total mass is prescribed:

$$\int_{\Omega} \varrho dx = M > 0.$$

We impose that the velocity \mathbf{u} satisfies no-slip boundary condition and the temperature Θ is a constant ϑ_0 on the boundary of Ω , i.e.,

$$\mathbf{u} = 0, \quad \Theta = \vartheta_0 \quad \text{on } \partial\Omega. \quad (1.2)$$

In the last decades, the steady compressible heat-conductive Navier-Stokes equations have been studied by many mathematicians and there are a lot of results on the existence in the literature, here we recall some of them for both small and large external forces which are related to our study in this paper, and we refer to the monograph [20] for more details. When external forces are sufficiently small, Matsumura and Nishida in 1982/83 proved the existence of a solution with potential forces near a rest state [14, 15], while Valli and Zajackowski [23, 25] used the existence of global non-stationary solutions to get the existence of stationary solutions. Later, Valli [24] showed the existence of stationary solutions in the general case by using an idea of Padula [21] to decompose the equations into two parts that are governed by the Stokes equations and a transport equation, respectively. Beirão da Veiga [1] obtained more general existence results in the L^p -setting by decomposing the equations into three parts that are governed by the Stokes equations, a transport equation and Laplace's equation, respectively. In 1989, Farwig [5] showed the existence of solutions to the steady compressible heat-conductive Navier-Stokes equations for small forces with slip boundary condition.

When external forces are of arbitrary size, the existence of strong solutions was proved in [19, 16] for the case of potential forces. When the equations of state and the viscosity coefficients satisfy certain (growth) conditions, Novotný and Pokorný [17, 18] showed that weak or strong solutions to the steady compressible heat-conductive Navier-Stokes equations exist. Unfortunately, their results exclude the case of ideal polytropic gases, for which the existence of strong solutions, to our best knowledge, still remains open.

The aim of the present paper is to establish the existence of strong solutions to the steady compressible heat-conductive Navier-Stokes system (1.1) without any smallness assumption on the external forces \mathbf{f} and \mathbf{g} , when the Mach number is small.

We mention that in the isentropic flow case, the existence of weak solutions or strong solutions under small Mach number for large external forces has been extensively investigated. Lions [13] first proved the existence of weak solutions under the assumption that the specific heat ratio $\gamma > 1$ in two dimensions and $\gamma > 5/3$ in three dimensions. The restriction on γ actually comes from the integrability of the density ϱ in L^p , and in fact, the higher integrability of ϱ has, the smaller γ can be allowed. In [20] Novotný and Straškraba showed the existence of weak solutions for any $\gamma > 3/2$ if \mathbf{f} is potential and $\mathbf{g} = 0$. By deriving a new weighted estimate of the pressure, Frehse, Goj and Steinhauer [6], Plotnikov and Sokolowski [22] established an improved integrability for the density under the assumption of the L^1 -boundedness of $\varrho \mathbf{u}^2$ which was not shown to hold unfortunately. In 2008, Březina and Novotný [3] was able to prove the existence of weak solution to the spatially periodic problem for any $\gamma > (3 + \sqrt{41})/8$ when \mathbf{f} is potential and $\mathbf{g} = 0$, or for any $\gamma > (1 + \sqrt{13})/3 \approx 1.53$ when $\mathbf{f}, \mathbf{g} \in L^\infty$, without assuming the L^1 -boundedness of $\varrho \mathbf{u}^2$, by combining the L^∞ -estimate of $\Delta^{-1}P$ with the (usual) energy and density bounds. Then, in the framework of [3], Frehse, Steinhauer and Weigant [7, 8] established the existence of weak solutions to the Dirichlet boundary value problem for any $\gamma > 4/3$ in three dimensions and to the spatially periodic or mixed boundary value problem for $\gamma = 1$ (isothermal flow) in two dimensions. Recently, Jiang and Zhou [10, 11] proved the existence of weak solutions to the spatially periodic or Dirichlet boundary value problem in \mathbb{R}^3 for any $\gamma > 1$. The existence of

strong solutions was shown by Choe and Jin [4] when the Mach number is small, by exploiting the known results for the incompressible steady Navier-Stokes equations.

Now, we rewrite (1.1) in the form of the Mach number. After scaling and a straightforward calculation we obtain the following dimensionless form of the steady full compressible Navier-Stokes equations:

$$\begin{cases} \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \varrho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\epsilon^2} = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} + \mathbf{g}, \\ \varrho \mathbf{u} \cdot \nabla \Theta + p \operatorname{div} \mathbf{u} = \kappa \Delta \Theta + \epsilon^2 \Psi, \end{cases} \quad (1.3)$$

where ϵ is the Mach number.

Since the total mass of the fluid is given, we impose the condition

$$\bar{\varrho} := \frac{1}{|\Omega|} \int_{\Omega} \varrho(x) dx > 0,$$

which can be renormalized to $\bar{\varrho} = 1$ without loss of generality. Similarly, we also assume that $\bar{\Theta} = 1$, $R = c_v = 1$, $\vartheta_0 = 1$.

To show the existence, we take the transformation

$$\varrho = 1 + \epsilon \rho, \quad \Theta = 1 + \epsilon \theta \quad (1.4)$$

to rewrite the system (1.3) in the form:

$$\begin{cases} \operatorname{div} \mathbf{u} + \epsilon \operatorname{div}(\rho \mathbf{u}) = 0, \\ (1 + \epsilon \rho)(\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{(1 + \epsilon \theta) \nabla \rho}{\epsilon} + \frac{(1 + \epsilon \rho) \nabla \theta}{\epsilon} = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + (1 + \epsilon \rho) \mathbf{f} + \mathbf{g}, \\ \epsilon(1 + \epsilon \rho) \mathbf{u} \cdot \nabla \theta + \operatorname{div} \mathbf{u} + (\epsilon \rho + \epsilon \theta + \epsilon^2 \rho \theta) \operatorname{div} \mathbf{u} = \epsilon \kappa \Delta \theta + \epsilon^2 \Psi, \end{cases} \quad (1.5)$$

with boundary conditions

$$\mathbf{u} = 0, \quad \theta = 0 \quad \text{on } \partial\Omega. \quad (1.6)$$

Now, we state the main result of this paper.

Theorem 1.1. *Let $\mathbf{f}, \mathbf{g} \in H^2(\Omega)$. Then there is an ϵ_0 depending on $\|(\mathbf{f}, \mathbf{g})\|_{H^2}$ and Ω , such that for any $\epsilon \in (0, \epsilon_0)$, there exists a solution $(\rho^\epsilon, \mathbf{u}^\epsilon, \theta^\epsilon) \in \bar{H}^2 \times (H^3 \cap H_0^1) \times (H^3 \cap H_0^1)$ to the boundary value problem (1.5), (1.6), satisfying*

$$\lim_{\epsilon \rightarrow 0} \inf_{U, P \in \mathbf{L}} \|\mathbf{u}^\epsilon - U\|_3 + \|\rho^\epsilon\|_2 + \|\theta^\epsilon\|_3 + \left\| \frac{\rho^\epsilon + \theta^\epsilon}{\epsilon} - P \right\|_2 = 0,$$

where $(U, P) \in \mathbf{L} := \{(U, P) \in (H^4 \cap H_0^1) \times \bar{H}^3 \mid (U, P) \text{ is a solution of the incompressible steady Navier-Stokes equations (1.7) with external force } \mathbf{f} + \mathbf{g}\}$, i.e.,

$$\begin{cases} U \cdot \nabla U - \mu \Delta U + \nabla P = \mathbf{f} + \mathbf{g}, \\ \operatorname{div} U = 0, \\ U = 0 \quad \text{on } \partial\Omega, \quad \int_{\Omega} P dx = 0. \end{cases} \quad (1.7)$$

Remark 1.1. *If considering the existence of spatially periodic solutions to (1.5) in a periodic domain, we can also obtain an existence result similar to Theorem 1.1.*

The system (1.5) is complicated mixed-type nonlinear equations containing such structures as elliptic and hyperbolic systems, for which the usual approach of the fixed point arguments used to prove the existence of classical solutions requires the smallness of data. To show Theorem 1.1 (the existence for large data), we split the system (1.5) into two parts, one of which is

similar to the steady incompressible Navier-Stokes equations with large force $\mathbf{f} + \mathbf{g}$, while another part corresponds to the steady compressible heat-conductive Navier-Stokes equations with small force $\epsilon \mathbf{f}$, provided the Mach number ϵ is small. Then, as noted in [4], we modify and combine elaborately the arguments in [9] where the existence of strong solutions to the incompressible Navier-Stokes equations for large forces was presented, and in [5] where strong solutions of the compressible viscous heat-conductive equations with small forces were dealt with, to establish Theorem 1.1.

Compared with the isentropic case studied in [4], due to presence of the energy equation, the main difficulties here lie in the existence of weak solutions to the approximate linearized system, dealing with the coupling terms between the velocity, density and temperature, and deriving the uniform estimates in a bounded domain, for example, how to control the energy norm $\|\mathbf{u} - U\|_3 + \|\eta\|_2 + \|\theta\|_3$ uniformly in ϵ under the no-slip boundary condition. To circumvent such difficulties, we take the transform of $\varrho = 1 + \epsilon\rho$, $\Theta = 1 + \epsilon\theta$ for the system (1.3), instead of the transform ($\varrho = 1 + \epsilon^2\rho$, $\Theta = 1 + \epsilon^2\theta$) used in [4], and utilize the lower order terms to control the higher order terms. More precisely, the main steps of the proof are the following: First, we apply Lax-Milgram's theorem to get the existence of weak solutions to the regularized elliptic equations (2.14)–(2.16) of the transformed linearized equations. Then, we exploit the uniform estimates and a compactness argument to get the existence of a weak solution to the transformed linearized system (2.12). We can easily verify that the weak solution to the transformed linearized system (2.12) is indeed a weak solution to the linearized system (2.10). And this fact together with the uniqueness of weak solutions to the system (2.10) gives the existence of a weak solution to the approximate linearized equations (2.10) in Section 2.2. Second, we exploit the property of the momentum equations and the regularity of the Stokes problem to establish the estimates of $|\eta + \theta|$, which, combined with an estimate for θ , implies the boundedness for η . Due to presence of boundary here, some difficulties involved with controlling the boundary terms arise. To overcome such difficulties, the crucial step is to get a H^2 -bound of $\text{div} \mathbf{u}$ near the boundary, for which we shall adopt the local isothermal coordinates used in [23, 25]. This strategy has also been used in [2, 12] to study the low Mach limit of the compressible Navier-Stokes with non-slip boundary condition. Then, summing up all the estimates for (\mathbf{v}, θ) and η , we can establish the desired a priori uniform in ϵ estimates in view of the smallness of ϵ (see Section 3). Finally, we apply the Tikhonov fixed point theorem to obtain the existence of a strong solution. Moreover, with the help of the uniform a priori estimates, one can take to the limit to show the incompressible limit. We point out that due to the splitting, we have to impose that the energy equation should not possess a heat source. Furthermore, it is still open whether a strong solution of the steady incompressible or compressible Navier-Stokes equations is unique.

The paper is organized as follows. In the next section, we will prove the existence of weak solutions and regularity to the linearized incompressible and compressible problems. Section 3 is devoted to establishing the existence for the nonlinear problem. Finally, the incompressible limit of solutions to the steady compressible heat-conductive Navier-Stokes equations is presented in Section 4.

Notations: We denote by L^2 the Lebesgue space $L^2(\Omega)$ with norm $\|\cdot\|_0$, by H^m the Sobolev spaces $H^m(\Omega)$ with norm $\|\cdot\|_m$. Define the spaces

$$\bar{H}^m = \left\{ \rho \in H^m \mid \int_{\Omega} \rho(x) dx = 0 \right\}, \quad H_{0,\sigma}^1 = \left\{ \mathbf{u} \in H_0^1 \mid \text{div} \mathbf{u} = 0 \right\}, \quad H_{0,\sigma}^m := H^m \cap H_{0,\sigma}^1.$$

We denote by H^{-1} the dual space of H_0^1 with the dual product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_{-1} = \sup_{\|h\|_1=1} |\langle \cdot, h \rangle|$. We shall use the abbreviation:

$$\int \cdot dx := \int_{\Omega} \cdot dx.$$

2 Existence of solutions to the linearized problem

We first split the system (1.5) into two parts, so that one part looks like the incompressible Navier-Stokes equations, while the other part behaves like the compressible Navier-Stokes equations. More precisely, let (U, P) and (v, η) be the solutions to the following systems, respectively:

$$\begin{cases} U \cdot \nabla U + \mathbf{v} \cdot \nabla U - \mu \Delta U + \nabla P = \mathbf{f} + \mathbf{g}, & \int_{\Omega} P = 0, \\ \operatorname{div} U = 0, \\ U = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_{\Omega} P = 0; \end{cases} \quad (2.1)$$

and

$$\begin{cases} U \cdot \nabla \eta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} = -\mathbf{v} \cdot \nabla \eta - \eta \operatorname{div} \mathbf{v} - \epsilon \operatorname{div}(P(U + \mathbf{v})), \\ U \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{v} + \frac{\nabla \eta + \nabla \theta}{\epsilon} = \epsilon F - \mathbf{v} \cdot \nabla \mathbf{v} - \theta \nabla \eta - \eta \nabla \theta, \\ U \cdot \nabla \theta - \kappa \Delta \theta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} = \epsilon G - \mathbf{v} \cdot \nabla \theta - \eta \operatorname{div} \mathbf{v} - \theta \operatorname{div} \mathbf{v}, \\ \mathbf{v} = 0, \quad \theta = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_{\Omega} \eta = 0, \end{cases} \quad (2.2)$$

where the new force F and heat source G are defined by

$$\begin{aligned} F &= (\epsilon P + \eta) \mathbf{f} - (\epsilon P + \eta)(U + \mathbf{v}) \cdot \nabla(U + \mathbf{v}) - \theta \nabla P - P \nabla \theta, \\ G &= \Psi - (\epsilon P + \eta)(U + \mathbf{v}) \cdot \nabla \theta - (\epsilon P + \eta) \theta \operatorname{div} \mathbf{v} - P \operatorname{div} \mathbf{v}. \end{aligned}$$

It is clear to observe that $\mathbf{u} := U + \mathbf{v}$, $\rho := \epsilon P + \eta$ and θ are a solution to (1.5). Thus, we can obtain a solution of the system (1.5) if we can solve the systems (2.1) and (2.2). First, we will give the existence of weak solutions to the linearized incompressible problem (2.1) and derive a priori estimates of higher order derivatives of the unknowns (U, P) . Then, we shall show the existence of weak solutions to the linearized compressible problem (2.2) and establish uniform estimates of higher order derivatives of the unknowns $(\eta, \mathbf{v}, \theta)$.

In what follows, we assume that $\operatorname{meas}(\Omega) = 1$ without loss of generality.

2.1 Linearized incompressible equations

Let \tilde{U} and $\tilde{\mathbf{v}}$ be given functions satisfying $\tilde{U} \in H^4 \cap H_{0,\sigma}^1$ and $\tilde{\mathbf{v}} \in H^3 \cap H_0^1$. At first, we consider the linearized equations to (2.1) for given \tilde{U} and $\tilde{\mathbf{v}}$ as follows.

$$\begin{cases} (\tilde{U} + \tilde{\mathbf{v}}) \cdot \nabla U - \mu \Delta U + \nabla P = \mathbf{h}, & \int_{\mathbb{T}} P = 0, \\ \operatorname{div} U = 0, \\ U = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_{\Omega} P = 0. \end{cases} \quad (2.3)$$

where $\mathbf{h} = \mathbf{f} + \mathbf{g}$.

The problem (2.3) is a Stokes problem which is solvable for arbitrarily large forces. In fact, (2.3) can be solved by using the Lax-Milgram theorem for small $\tilde{\mathbf{v}}$, and we can obtain the following existence result, the proof of which can be found, for example, in [4], and is therefore omitted here.

Lemma 2.1. *Let $\mathbf{h} \in H^{-1}$. There exists a constant a_0 depending on μ and Ω , such that if $\|\tilde{\mathbf{v}}\|_3 < a_0$, then there exists a weak solution $(U, P) \in H_0^1 \times \bar{H}^0$ of (2.3), satisfying*

$$\|U\|_1 \leq C_0 \|\mathbf{h}\|_{-1}, \quad (2.4)$$

$$\|P\|_0 \leq C_1 \|\mathbf{h}\|_{-1} (1 + \|\mathbf{h}\|_{-1}), \quad (2.5)$$

where C_0 and C_1 are positive constants which depend only on Ω , μ and a_0 .

As for the regularity of solutions, we consider the Stokes equations:

$$\begin{aligned} -\mu \Delta U + \nabla P &= \mathbf{h} - (\tilde{U} + \tilde{\mathbf{v}}) \cdot \nabla U, \\ \operatorname{div} U &= 0. \end{aligned}$$

Then we can derive the following estimates by employing bootstrap arguments similar to those in [4].

Lemma 2.2. *Let $\mathbf{h} \in H^m$, $\tilde{U} \in H^{m+1} \cap H_0^1$, $m = 0, 1, 2$, and $\tilde{\mathbf{v}}$ be the same as in Lemma 2.1. There are positive constants C_2 , C_3 and C_4 , depending only on Ω , μ and a_0 , such that if $\tilde{U} \in H_0^1$ satisfies the inequality (2.4), then*

$$\|U\|_2 + \|\nabla P\|_0 \leq C_2 \|\mathbf{h}\|_0 (\|\mathbf{h}\|_0 + 1)^4. \quad (2.6)$$

If $\tilde{U} \in H^2 \cap H_0^1$ satisfies (2.6), then

$$\|U\|_3 + \|\nabla P\|_1 \leq C_3 \|\mathbf{h}\|_1 (\|\mathbf{h}\|_1 + 1)^8, \quad (2.7)$$

and if $\tilde{U} \in H^3 \cap H_0^1$ satisfies (2.7), then

$$\|U\|_4 + \|\nabla P\|_2 \leq C_4 \|\mathbf{h}\|_2 (\|\mathbf{h}\|_2 + 1)^{12}. \quad (2.8)$$

Let $\mathbf{f}, \mathbf{g} \in H^2(\Omega)$, then it is obvious that $\mathbf{h} \in H^2(\Omega)$. We define a function space K_0 by

$$\begin{aligned} K_0 := \left\{ U \in H_{0,\sigma}^4(\Omega) : \|U\|_1 \leq C_1 \|\mathbf{h}\|_1, \|U\|_2 \leq C_2 \|\mathbf{h}\|_0 (\|\mathbf{h}\|_0 + 1)^4, \right. \\ \left. \|U\|_3 \leq C_3 \|\mathbf{h}\|_1 (\|\mathbf{h}\|_1 + 1)^8, \|U\|_4 \leq C_4 \|\mathbf{h}\|_2 (\|\mathbf{h}\|_2 + 1)^{12} \right\}. \end{aligned} \quad (2.9)$$

Thus, by Lemma 2.2 we see that the solution U of the system (2.3) also lies in K_0 for any given $\tilde{U} \in K_0$, since the constants C_1, \dots, C_4 do not depend on \tilde{U} .

2.2 Linearized compressible equations

Let $(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta}) \in (H^4 \cap H_{0,\sigma}^1) \times (H^3 \cap H_0^1) \times (H^3 \cap H_0^1)$ be given functions, and (U, P) be the solution of (2.3) established in Section 2.1. Next, we give the existence of weak solutions and derive some a priori estimates for solutions to the linearized equations of the system (2.2). For simplicity, we only consider the three-dimensional case. As aforementioned, we shall apply the Tikhonov fixed point theorem to show the existence of steady strong solutions to (1.5). To this end, for given $(\tilde{\mathbf{v}}, \tilde{\theta}) \in (H^3 \cap H_0^1) \times (H^3 \cap H_0^1)$, let $(\eta, \mathbf{v}, \theta)$ be the unique solution of the following linearized system of (2.2) the existence of which will be shown below:

$$\begin{cases} U \cdot \nabla \eta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} + \tilde{\mathbf{v}} \cdot \nabla \eta + \eta \operatorname{div} \tilde{\mathbf{v}} = -\epsilon \operatorname{div}(P(U + \tilde{\mathbf{v}})), \\ U \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} - \zeta \nabla \operatorname{div} \mathbf{v} + \frac{\nabla \eta + \nabla \theta}{\epsilon} + \tilde{\theta} \nabla \eta + \eta \nabla \tilde{\theta} = \epsilon \tilde{F} - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}, \\ U \cdot \nabla \theta - \kappa \Delta \theta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} + \eta \operatorname{div} \tilde{\mathbf{v}} = \epsilon \tilde{G} - \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} - \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}}, \end{cases} \quad (2.10)$$

where the new force \tilde{F} and heat source \tilde{G} are defined by

$$\begin{aligned} \tilde{F} &= (\epsilon P + \eta) \mathbf{f} - (\epsilon P + \eta)(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) - \tilde{\theta} \nabla P - P \nabla \tilde{\theta}, \\ \tilde{G} &= \tilde{\Psi} - (\epsilon P + \eta)(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} - (\epsilon P + \eta) \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} - P \operatorname{div} \tilde{\mathbf{v}}, \end{aligned}$$

with $\tilde{\Psi} = 2\mu D(U + \tilde{\mathbf{v}}) : D(U + \tilde{\mathbf{v}}) + \lambda(\operatorname{div}(U + \tilde{\mathbf{v}}))^2$ and $\zeta = \mu + \lambda$.

Thus, for given $\tilde{U}, \tilde{\mathbf{v}}$ and $\tilde{\theta}$, we can construct a map N :

$$N(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta}) := (U, \mathbf{v}, \theta).$$

And, we have to show that N maps some space into itself and is weak continuous to get a fixed point of the mapping N .

In order to obtain the existence of weak solution of (2.10), we set

$$\tilde{F}' = \epsilon P \mathbf{f} - \epsilon P(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) - \tilde{\theta} \nabla P - P \nabla \tilde{\theta},$$

$$\tilde{G}' = \tilde{\Psi} - \epsilon P(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} - \epsilon P \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} - P \operatorname{div} \tilde{\mathbf{v}}.$$

So, $\tilde{F} = \tilde{F}' + \eta \mathbf{f} + \eta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}})$, $\tilde{G} = \tilde{G}' - \eta(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} - \eta \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}}$.

2.2.1 Existence of weak solutions

Lemma 2.3. *Let $\tilde{F}', \tilde{G}' \in H^{-1}$, $\mathbf{f} \in H^2$, and (U, P) be a solution of (2.3) established in Lemma 2.1. If $\|\tilde{\mathbf{v}}\|_3 + \|\tilde{\theta}\|_3$ is sufficiently small, then there exists a unique weak solution $(\eta, \mathbf{v}, \theta) \in \tilde{H}^0 \times H_0^1 \times H_0^1$ to the equations (2.10) with boundary condition (2.2)₄.*

Proof. First, the momentum equations (2.10)₂ can be rewritten as

$$\begin{aligned} \eta + \theta + \epsilon \eta \tilde{\theta} + \epsilon \Delta^{-1} \operatorname{div}(U \cdot \nabla \mathbf{v}) - \epsilon(\mu + \zeta) \operatorname{div} \mathbf{v} + \epsilon^2 \Delta^{-1} \operatorname{div} \left[\eta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] \\ - \epsilon^2 \Delta^{-1} \operatorname{div}(\eta \mathbf{f}) = \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}). \end{aligned} \quad (2.11)$$

We add (2.11) to (2.10)₁ and (2.10)₃, respectively, and rewrite the system (2.10) as the following equations:

$$\left\{ \begin{aligned} & U \cdot \nabla \eta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} + \tilde{\mathbf{v}} \cdot \nabla \eta + \eta \operatorname{div} \tilde{\mathbf{v}} + \eta + \theta + \epsilon \eta \tilde{\theta} + \epsilon \Delta^{-1} \operatorname{div}(U \cdot \nabla \mathbf{v}) \\ & \quad - \epsilon(\mu + \zeta) \operatorname{div} \mathbf{v} + \epsilon^2 \Delta^{-1} \operatorname{div} \left[\eta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] - \epsilon^2 \Delta^{-1} \operatorname{div}(\eta \mathbf{f}) \\ & \quad = -\epsilon \operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}), \\ & U \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} - \zeta \nabla \operatorname{div} \mathbf{v} + \frac{\nabla \eta + \nabla \theta}{\epsilon} + \tilde{\theta} \nabla \eta + \eta \nabla \tilde{\theta} - \epsilon \eta \mathbf{f} + \epsilon \eta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \\ & \quad = \epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}, \\ & U \cdot \nabla \theta - \kappa \Delta \theta + \frac{\operatorname{div} \mathbf{v}}{\epsilon} + \eta \operatorname{div} \tilde{\mathbf{v}} + \epsilon \eta(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} - \epsilon \eta \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} + \eta + \theta + \epsilon \eta \tilde{\theta} + \epsilon \Delta^{-1} \operatorname{div}(U \cdot \nabla \mathbf{v}) \\ & \quad - \epsilon(\mu + \zeta) \operatorname{div} \mathbf{v} + \epsilon^2 \Delta^{-1} \operatorname{div} \left[\eta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] - \epsilon^2 \Delta^{-1} \operatorname{div}(\eta \mathbf{f}) \\ & \quad = \epsilon \tilde{G}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} - \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}), \end{aligned} \right. \quad (2.12)$$

with boundary conditions

$$\mathbf{v} = \theta = 0 \quad \text{on } \partial\Omega. \quad (2.13)$$

Consider the regularized elliptic equations of (2.12) as follows:

$$\begin{aligned} -\delta \Delta \eta^\delta + U \cdot \nabla \eta^\delta + \frac{\operatorname{div} \mathbf{v}^\delta}{\epsilon} + \tilde{\mathbf{v}} \cdot \nabla \eta^\delta + \eta^\delta \operatorname{div} \tilde{\mathbf{v}} + \eta^\delta + \theta^\delta + \epsilon \eta^\delta \tilde{\theta} + \epsilon \Delta^{-1} \operatorname{div}(U \cdot \nabla \mathbf{v}^\delta) \\ - \epsilon(\mu + \zeta) \operatorname{div} \mathbf{v}^\delta + \epsilon^2 \Delta^{-1} \operatorname{div} \left[\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] - \epsilon^2 \Delta^{-1} \operatorname{div}(\eta^\delta \mathbf{f}) \\ = -\epsilon \operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}'^\delta - \tilde{\mathbf{v}}^\delta \cdot \nabla \tilde{\mathbf{v}}), \end{aligned} \quad (2.14)$$

$$U \cdot \nabla \mathbf{v}^\delta - \mu \Delta \mathbf{v}^\delta - \zeta \nabla \operatorname{div} \mathbf{v}^\delta + \frac{\nabla \eta^\delta + \nabla \theta^\delta}{\epsilon} + \tilde{\theta}^\delta \nabla \eta^\delta + \eta^\delta \nabla \tilde{\theta}^\delta - \epsilon \eta^\delta \mathbf{f}$$

$$+\epsilon\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) = \epsilon\tilde{F}'^\delta - \tilde{\mathbf{v}}^\delta \cdot \nabla\tilde{\mathbf{v}}, \quad (2.15)$$

$$\begin{aligned} & U \cdot \nabla\theta^\delta - \kappa\Delta\theta^\delta + \frac{\operatorname{div}\mathbf{v}^\delta}{\epsilon} + \eta^\delta\operatorname{div}\tilde{\mathbf{v}} + \epsilon\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla\tilde{\theta} - \epsilon\eta^\delta\tilde{\theta}\operatorname{div}\tilde{\mathbf{v}} + \eta^\delta + \theta^\delta + \epsilon\eta^\delta\tilde{\theta} \\ & + \epsilon\Delta^{-1}\operatorname{div}(U \cdot \nabla\mathbf{v}^\delta) - \epsilon(\mu + \zeta)\operatorname{div}\mathbf{v}^\delta + \epsilon^2\Delta^{-1}\operatorname{div}\left[\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}})\right] \\ & - \epsilon^2\Delta^{-1}\operatorname{div}(\eta^\delta\mathbf{f}) = \epsilon\tilde{G}' - \tilde{\mathbf{v}}^\delta \cdot \nabla\tilde{\theta} - \tilde{\theta}^\delta\operatorname{div}\tilde{\mathbf{v}} + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}}), \end{aligned} \quad (2.16)$$

with boundary conditions

$$\frac{\partial\eta^\delta}{\partial\mathbf{n}} = 0, \quad \mathbf{v}^\delta = \theta^\delta = 0 \quad \text{on } \partial\Omega. \quad (2.17)$$

Here \mathbf{n} is the outer normal vector.

The system (2.14)–(2.17) in variational form reads as: Find $(\eta^\delta, \mathbf{v}^\delta, \theta^\delta) \in \bar{H}^1 \times H_0^1 \times H_0^1$, such that

$$\begin{aligned} & B(\eta^\delta, \mathbf{v}^\delta, \theta^\delta; \underline{\eta}, \underline{\mathbf{v}}, \underline{\theta}) \\ & = \int [-\epsilon\operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}'^\delta - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})]\underline{\eta} + (\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})\underline{\mathbf{v}} \\ & + [\epsilon\tilde{G}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\theta} - \tilde{\theta}^\delta\operatorname{div}\tilde{\mathbf{v}} + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})]\underline{\theta}dx, \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} B(\eta^\delta, \mathbf{v}^\delta, \theta^\delta; \underline{\eta}, \underline{\mathbf{v}}, \underline{\theta}) := & \delta \int \nabla\eta^\delta \cdot \nabla\underline{\eta}dx - \int (U \cdot \nabla\underline{\eta})\eta^\delta + (U \cdot \nabla\underline{\mathbf{v}}) \cdot \mathbf{v}^\delta + (U \cdot \nabla\underline{\theta})\theta^\delta dx \\ & - \int (\tilde{\mathbf{v}} \cdot \nabla\underline{\eta})\eta^\delta dx + \int (\eta^\delta + \theta^\delta)(\underline{\eta} + \underline{\theta}) + \mu\nabla\mathbf{v}^\delta : \nabla\underline{\mathbf{v}} + \zeta(\operatorname{div}\mathbf{v}^\delta)(\operatorname{div}\underline{\mathbf{v}}) \\ & + \kappa\nabla\theta^\delta \cdot \nabla\underline{\theta}dx + \int \left\{ \epsilon\eta^\delta\tilde{\theta} + \epsilon\Delta^{-1}\operatorname{div}(U \cdot \nabla\mathbf{v}^\delta) - \epsilon(\mu + \zeta)\operatorname{div}\mathbf{v}^\delta \right. \\ & \left. - \epsilon^2\Delta^{-1}\operatorname{div}(\eta^\delta\mathbf{f}) + \epsilon^2\Delta^{-1}\operatorname{div}\left[\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}})\right] \right\}(\underline{\eta} + \underline{\theta})dx \\ & + \int \left[\epsilon\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) - \epsilon\eta^\delta\mathbf{f} \right] \cdot \underline{\mathbf{v}} - \eta^\delta\tilde{\theta}\operatorname{div}\underline{\mathbf{v}}dx \\ & + \int \left[\epsilon\eta^\delta(U + \tilde{\mathbf{v}}) \cdot \nabla\tilde{\theta} - \epsilon\eta^\delta\tilde{\theta}\operatorname{div}\tilde{\mathbf{v}} + \eta^\delta\operatorname{div}\tilde{\mathbf{v}} \right]\underline{\theta}dx \\ & + \frac{1}{\epsilon} \int (\operatorname{div}\mathbf{v}^\delta)\underline{\eta} - \operatorname{div}\underline{\mathbf{v}}(\eta^\delta + \theta^\delta) + (\operatorname{div}\mathbf{v}^\delta)\underline{\theta}dx, \end{aligned} \quad (2.19)$$

for any $(\underline{\eta}, \underline{\mathbf{v}}, \underline{\theta}) \in \bar{H}^1 \times H_0^1 \times H_0^1$.

We shall apply Lax-Milgram's theorem to show the solvability of the variational problem (2.18). We first show the coerciveness of the bilinear form in (2.19). To this end, we take $\eta^\delta = \underline{\eta}$, $\mathbf{v}^\delta = \underline{\mathbf{v}}$, $\theta^\delta = \underline{\theta}$ in (2.19), integrate by parts and use Poincaré's inequality $\|\theta^\delta\|_0 \leq \sqrt{c_0}\|\nabla\theta^\delta\|_0$ to see that

$$\begin{aligned} & B(\eta^\delta, \mathbf{v}^\delta, \theta^\delta; \eta^\delta, \mathbf{v}^\delta, \theta^\delta) \\ & \geq \int (\delta|\nabla\eta^\delta|^2 + (\eta^\delta + \theta^\delta)^2 + \mu|\nabla\mathbf{v}^\delta|^2 + \zeta(\operatorname{div}\mathbf{v}^\delta)^2 + \kappa|\nabla\theta^\delta|^2)dx - \int (\epsilon\|\tilde{\theta}\|_{L^\infty}|\eta^\delta|^2)dx \\ & - \epsilon\|U\|_3^2\|\nabla\mathbf{v}^\delta\|_0^2 - \|\tilde{\theta}\|_2\|\eta^\delta\|_0^2 - \|\tilde{\theta}\|_2\|\operatorname{div}\mathbf{v}^\delta\|_0^2 - \|\operatorname{div}\tilde{\mathbf{v}}\|_2(\|\eta^\delta\|_0^2 + \|\theta^\delta\|_0^2) \end{aligned}$$

$$\begin{aligned}
& -\epsilon(\|\mathbf{f}\|_2 + \|U + \tilde{\mathbf{v}}\|_3^2 + \|U + \tilde{\mathbf{v}}\|_2\|\tilde{\theta}\|_3 + \|\tilde{\mathbf{v}}\|_3\|\tilde{\theta}\|_2)\|\eta^\delta\|_0^2 \\
& -\epsilon(\|\mathbf{f}\|_2 + \|U + \tilde{\mathbf{v}}\|_3^2)\|\mathbf{v}^\delta\|_0^2 - \epsilon(\|U + \tilde{\mathbf{v}}\|_2\|\tilde{\theta}\|_3 + \|\tilde{\mathbf{v}}\|_3\|\tilde{\theta}\|_2)\|\theta^\delta\|_0^2 \\
& -\epsilon[\|\tilde{\theta}\|_2(\|\eta^\delta\|_0^2 + \|\theta^\delta\|_0^2) + \|\theta^\delta\|_0^2 + \|U\|_2^2\|\nabla\mathbf{v}^\delta\|_0^2 + (\mu + \zeta)^2\|\theta^\delta\|_0^2 + \|\nabla\mathbf{v}^\delta\|_0^2 \\
& + \|U + \tilde{\mathbf{v}}\|_3^2\|\eta^\delta\|_0^2 + \|\theta^\delta\|_0^2 + \epsilon\|\mathbf{f}\|_2(\|\eta^\delta\|_0^2 + \|\theta^\delta\|_0^2)] \\
& \geq \left[\frac{\kappa}{2c_0 + \kappa} - 2\epsilon\|\tilde{\theta}\|_2 - \epsilon(\mu + \zeta)^2 - \|\tilde{\theta}\|_2 - \|\operatorname{div}\tilde{\mathbf{v}}\|_2 - (\epsilon + \epsilon^2\sqrt{c_0})(2\|\mathbf{f}\|_2 \right. \\
& \quad \left. + 2\|U + \tilde{\mathbf{v}}\|_3^2 + \|U + \tilde{\mathbf{v}}\|_2\|\tilde{\theta}\|_3 + \|\tilde{\mathbf{v}}\|_3\|\tilde{\theta}\|_2) \right] \|\eta^\delta\|_0^2 + \delta\|\nabla\eta^\delta\|_0^2 \\
& \quad + \left[\mu - 2\epsilon\|U\|_3^2 - \epsilon c_0(\|\mathbf{f}\|_2 + \|U + \tilde{\mathbf{v}}\|_3^2) \right] \|\nabla\mathbf{v}^\delta\|_0^2 + (\zeta - \|\tilde{\theta}\|_2 - \epsilon)\|\operatorname{div}\mathbf{v}^\delta\|_0^2 \\
& \quad + \left[\frac{\kappa}{2} - c_0\|\operatorname{div}\tilde{\mathbf{v}}\|_2 - \epsilon c_0(\|U + \tilde{\mathbf{v}}\|_2\|\tilde{\theta}\|_3 + \|\tilde{\mathbf{v}}\|_3\|\tilde{\theta}\|_2 + \|\tilde{\theta}\|_2 \right. \\
& \quad \left. + 1 + (\mu + \zeta)^2 + \|\mathbf{f}\|_2) \right] \|\nabla\theta^\delta\|_0^2 \\
& \geq \delta\|\nabla\eta^\delta\|_0^2 + C'(\|\eta^\delta\|_0^2 + \|\mathbf{v}^\delta\|_1^2 + \|\theta^\delta\|_1^2), \tag{2.20}
\end{aligned}$$

where the constant C' depends only on Ω , μ , λ , κ , c_0 , $\|\tilde{\theta}\|_2$ and $\|\tilde{\mathbf{v}}\|_3$. Notice that here the smallness of ϵ , $\|\tilde{\theta}\|_2$ and $\|\tilde{\mathbf{v}}\|_3$ are necessary. In fact, we have used the following estimate in the second inequality of (2.20):

$$\begin{aligned}
\int ((\eta^\delta + \theta^\delta)^2 + \frac{\kappa}{2}|\nabla\theta^\delta|^2)dx & \geq \int ((\eta^\delta + \theta^\delta)^2 + \frac{\kappa}{2c_0}|\theta^\delta|^2)dx \\
& = \int \left((1 + \frac{\kappa}{2c_0})(\theta^\delta + \frac{2c_0}{2c_0 + \kappa}\eta^\delta)^2 + \frac{\kappa}{2c_0 + \kappa}|\eta^\delta|^2 \right)dx \\
& \geq \frac{\kappa}{2c_0 + \kappa}\|\eta^\delta\|_0^2.
\end{aligned}$$

Obviously, the bilinear form in (2.19) is continuous on $\bar{H}^1 \times H_0^1 \times H_0^1$. Thus, there is a unique solution $(\eta^\delta, \mathbf{v}^\delta, \theta^\delta)$ of (2.14)–(2.17) satisfying the variation form (2.18) for all $(\underline{\eta}, \underline{\mathbf{v}}, \underline{\theta}) \in \bar{H}^1 \times H_0^1 \times H_0^1$. Moreover, taking ϵ so small that $\epsilon < \frac{C'}{2}$, we obtain

$$\begin{aligned}
& \delta\|\nabla\eta^\delta\|_0^2 + C(\|\eta^\delta\|_0^2 + \|\mathbf{v}^\delta\|_1^2 + \|\theta^\delta\|_1^2) \\
& \leq \|\epsilon\operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})\|_0^2 + \|\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}}\|_0^2 \\
& \quad + \|\epsilon\tilde{G}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\theta} - \tilde{\theta}^\delta\operatorname{div}\tilde{\mathbf{v}} + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})\|_0^2
\end{aligned}$$

for some positive constant C .

By Rellich's compactness theorem, there is a subsequence $\eta^{\delta_n} \in \bar{H}^0(\Omega)$, $\mathbf{v}^{\delta_n} \in H_0^1(\Omega)$, $\theta^{\delta_n} \in H_0^1(\Omega)$, and $\eta \in \bar{H}^0(\Omega)$, $\mathbf{v} \in H_0^1(\Omega)$, $\theta \in H_0^1(\Omega)$ satisfying that

$$\eta^{\delta_n} \text{ converges weakly to } \eta \text{ in } \bar{H}^0(\Omega)$$

$$\mathbf{v}^{\delta_n}, \theta^{\delta_n} \text{ converges weakly to } \mathbf{v}, \theta \text{ in } H_0^1(\Omega)$$

and

$$\mathbf{v}^{\delta_n}, \theta^{\delta_n} \text{ converges strongly to } \mathbf{v}, \theta \text{ in } H^0(\Omega),$$

respectively, as $n \rightarrow \infty$. And it can be easily validated that $(\eta, \mathbf{v}, \theta)$ is indeed the unique weak solution of (2.12). Due to the lower semicontinuity of the H^m -norm ($m = 0, 1$) and the estimate

$$\begin{aligned}
& \delta\|\nabla\eta^{\delta_n}\|_0^2 + (\|\eta^{\delta_n}\|_0^2 + \|\mathbf{v}^{\delta_n}\|_1^2 + \|\theta^{\delta_n}\|_1^2) \\
& \leq \|\epsilon\operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon\Delta^{-1}\operatorname{div}(\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}})\|_0^2 + \|\epsilon\tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}}\|_0^2
\end{aligned}$$

$$+\|\epsilon\tilde{G}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} - \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}})\|_0^2,$$

we find that

$$\begin{aligned} & \|\eta\|_0^2 + \|\mathbf{v}\|_1^2 + \|\theta\|_1^2 \\ & \leq \|\epsilon \operatorname{div}(P(U + \tilde{\mathbf{v}})) + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}})\|_0^2 + \|\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}\|_0^2 \\ & \quad + \|\epsilon \tilde{G}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} - \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} + \epsilon \Delta^{-1} \operatorname{div}(\epsilon \tilde{F}' - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}})\|_0^2. \end{aligned}$$

On the other hand, it is easy to verify that a weak solution of (2.12) is also a weak solution of (2.10). Moreover, a weak solution of (2.10) is unique. In fact, assuming that $(\eta_1, \mathbf{v}_1, \theta_1), (\eta_2, \mathbf{v}_2, \theta_2) \in \bar{H}^0 \times H_0^1 \times H_0^1$ are two weak solutions to the problem (2.10) with boundary condition (2.2)₄, and letting $\bar{\eta} = \eta_1 - \eta_2$, $\bar{\mathbf{v}} = \mathbf{v}_1 - \mathbf{v}_2$, $\bar{\theta} = \theta_1 - \theta_2$, we find that $(\bar{\eta}, \bar{\mathbf{v}}, \bar{\theta})$ is a weak solution of the following boundary value problem:

$$\begin{cases} U \cdot \nabla \bar{\eta} + \frac{\operatorname{div} \bar{\mathbf{v}}}{\epsilon} + \tilde{\mathbf{v}} \cdot \nabla \bar{\eta} + \bar{\eta} \operatorname{div} \tilde{\mathbf{v}} = 0, \\ U \cdot \nabla \bar{\mathbf{v}} - \mu \Delta \bar{\mathbf{v}} - \zeta \nabla \operatorname{div} \bar{\mathbf{v}} + \frac{\nabla \bar{\eta} + \nabla \bar{\theta}}{\epsilon} + \tilde{\theta} \nabla \bar{\eta} + \bar{\eta} \nabla \tilde{\theta} = \epsilon(\bar{\eta} \mathbf{f} + \bar{\eta}(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}})), \\ U \cdot \nabla \bar{\theta} - \kappa \Delta \bar{\theta} + \frac{\operatorname{div} \bar{\mathbf{v}}}{\epsilon} + \bar{\eta} \operatorname{div} \tilde{\mathbf{v}} = \epsilon(-\bar{\eta}(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} - \bar{\eta} \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}}), \\ \bar{\mathbf{v}} = 0, \quad \bar{\theta} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_{\Omega} \bar{\eta} = 0. \end{cases} \quad (2.21)$$

Then, we test the equations (2.21)₁, (2.21)₂, (2.21)₃ with $\bar{\eta}, \bar{\mathbf{v}}, \bar{\theta}$ respectively to deduce that

$$\begin{aligned} & \int \mu |\nabla \bar{\mathbf{v}}|^2 + \zeta |\nabla \bar{\mathbf{v}}|^2 dx + \int \kappa |\nabla \bar{\theta}|^2 dx \\ & = \epsilon \int \bar{\eta} \bar{\mathbf{v}} \left[\mathbf{f} + (U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] - \bar{\eta} \bar{\theta} \left[(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} + \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} \right] dx \\ & \quad - \int \frac{1}{2} \operatorname{div} \tilde{\mathbf{v}} |\bar{\eta}|^2 + \bar{\eta} \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} \bar{\theta} + \bar{\eta} \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} dx \\ & \leq (\|\bar{\eta}\|_0^2 + \|\bar{\theta}\|_0^2) \left[(\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\tilde{\theta}\|_3 + \|\tilde{\mathbf{v}}\|_3 (1 + \|\tilde{\theta}\|_2) \right] \\ & \quad + \epsilon C (\|\bar{\eta}\|_0^2 + \|\bar{\mathbf{v}}\|_0^2) (\|\mathbf{f}\|_2 + \|U\|_2 + \|\tilde{\mathbf{v}}\|_2). \end{aligned} \quad (2.22)$$

And according to the inhomogeneous Stokes equations:

$$\begin{cases} -\mu \Delta \bar{\mathbf{v}} + \frac{\nabla \bar{\eta} + \nabla \bar{\theta}}{\epsilon} = -U \cdot \nabla \bar{\mathbf{v}} + \zeta \nabla \operatorname{div} \bar{\mathbf{v}} + \epsilon \left[\bar{\eta} \mathbf{f} + \bar{\eta}(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) \right] - (\tilde{\theta} \nabla \bar{\eta} + \bar{\eta} \nabla \tilde{\theta}), \\ \operatorname{div} \bar{\mathbf{v}} = \operatorname{div} \tilde{\mathbf{v}}, \\ \bar{\mathbf{v}} = 0, \end{cases}$$

we have

$$\begin{aligned} & \epsilon \|\bar{\mathbf{v}}\|_1 + \|\bar{\eta} + \bar{\theta}\|_0 \\ & \leq \epsilon C \left\{ \|U\|_2 \|\bar{\mathbf{v}}\|_1 + \|\bar{\mathbf{v}}\|_1 + \epsilon \|\bar{\eta}\|_0 \left[\|\mathbf{f}\|_2 + (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2)^2 \right] + \|\bar{\eta}\|_0 \|\tilde{\theta}\|_2 \right\}. \end{aligned} \quad (2.23)$$

By combining (2.22) and (2.23), making use of Poincaré's inequality and the smallness of ϵ , $\|\tilde{\mathbf{v}}\|_3$ and $\|\tilde{\theta}\|_3$, we see that

$$\|\bar{\eta}\|_0 + \|\bar{\mathbf{v}}\|_1 + \|\bar{\theta}\|_1 \leq 0, \quad (2.24)$$

which implies $\eta_1 = \eta_2$, $\mathbf{v}_1 = \mathbf{v}_2$, $\theta_1 = \theta_2$. Hence, there is a unique weak solution $(\eta, \mathbf{v}, \theta) \in \bar{H}^0 \times H_0^1 \times H_0^1$ of the problem (2.10). \square

In order to get higher order uniform in ϵ estimates of the $(\eta, \mathbf{v}, \theta)$, we give another bound about $\|\mathbf{v}\|_1$ and $\|\theta\|_1$.

Lemma 2.4. *Let (U, P) be the solution of (2.3) given in Lemma 2.1. Let $\tilde{F}, \tilde{G} \in H^{-1}$. Then we have the following uniform in ϵ estimate:*

$$\begin{aligned} \|\mathbf{v}\|_1^2 + \|\theta\|_1^2 \leq & C_5 \left[\|\tilde{\mathbf{v}}\|_3 \|\eta\|_0^2 + \epsilon^2 (\|\tilde{F}\|_{-1}^2 + \|\tilde{G}\|_{-1}^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_0 \right. \\ & \left. + \|\tilde{\mathbf{v}}\|_1^4 + \|\eta\|_1^2 \|\tilde{\theta}\|_1^2 + \|\tilde{\mathbf{v}}\|_1^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2) \right], \end{aligned} \quad (2.25)$$

where the constant $C_5 > 0$ is independent of ϵ .

Proof. Multiplying (2.10)₁, (2.10)₂ and (2.10)₃ by η , v and θ in L^2 respectively, and summing up the resulting equations, we find that

$$\begin{aligned} & C' (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) \\ &= -\frac{1}{\epsilon} \int (\operatorname{div} \mathbf{v} (\eta + \theta) + v \cdot (\nabla \eta + \nabla \theta)) dx - \int [(U + \tilde{\mathbf{v}}) \cdot \nabla \eta \cdot \eta + \eta^2 \operatorname{div} \tilde{\mathbf{v}} + \epsilon \operatorname{div} (P(U + \tilde{\mathbf{v}})) \eta] dx \\ & \quad + \int (\epsilon \tilde{F} - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} - \tilde{\theta} \nabla \eta - \eta \nabla \tilde{\theta}) \cdot \mathbf{v} dx + \int (\epsilon \tilde{G} - \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} - (\eta + \tilde{\theta}) \operatorname{div} \tilde{\mathbf{v}}) \theta dx \\ &\leq \frac{1}{2} \|\tilde{\mathbf{v}}\|_3 \|\eta\|_0^2 + \delta (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) + C_\delta [\epsilon^2 (\|\tilde{F}\|_{-1}^2 + \|\tilde{G}\|_{-1}^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_0 \\ & \quad + \|\tilde{\mathbf{v}}\|_1^4 + \|\eta\|_1^2 \|\tilde{\theta}\|_1^2 + \|\tilde{\mathbf{v}}\|_1^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2)], \end{aligned} \quad (2.26)$$

where we have used integration by parts, Sobolev's inequality and the fact that

$$-\frac{1}{\epsilon} \int ((\eta + \theta) \operatorname{div} \mathbf{v} + v \cdot (\nabla \eta + \nabla \theta)) dx = \frac{1}{\epsilon} \int [(\eta + \theta) \operatorname{div} \mathbf{v} - (\eta + \theta) \operatorname{div} \mathbf{v}] dx = 0.$$

Finally, if we take δ in (2.26) suitably small and apply Poincaré's inequality, we obtain the estimate (2.25). \square

2.2.2 Stokes problem

We rewrite the momentum equations (2.10)₂ as an inhomogeneous Stokes problem to derive the desired bounds for $\|\mathbf{v}\|_3$ and $\|\frac{\nabla(\eta + \theta)}{\epsilon}\|_1$:

$$\begin{cases} -\mu \Delta \mathbf{v} + \frac{\nabla \eta + \nabla \theta}{\epsilon} = \epsilon \tilde{F} - \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} - \tilde{\theta} \nabla \eta - \eta \nabla \tilde{\theta} - U \cdot \nabla \mathbf{v} + \zeta \nabla \operatorname{div} \mathbf{v}, \\ \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{v}, \\ \mathbf{v} = 0, \quad \text{on } \Omega. \end{cases} \quad (2.27)$$

By the usual estimates for the steady Stokes problem (cf. Galdi's book [9, Chapter IV]), Sobolev's embedding $H^2 \hookrightarrow L^\infty$ and the inequality

$$\|\operatorname{div} \mathbf{v}\|_1^2 \leq \delta \|\mathbf{v}\|_3^2 + C_\delta \|\mathbf{v}\|_1^2, \quad (2.28)$$

we have

$$\begin{aligned} \|\mathbf{v}\|_2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0 \leq & C (\|\epsilon \tilde{F}\|_0 + \|\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}\|_0 + \|\tilde{\theta} \nabla \eta\|_0 + \|\eta \nabla \tilde{\theta}\|_0 \\ & + \|U \cdot \nabla \mathbf{v}\|_0 + \|\operatorname{div} \mathbf{v}\|_1). \end{aligned} \quad (2.29)$$

and

$$\|\mathbf{v}\|_3 + \left\| \frac{\nabla\eta + \nabla\theta}{\epsilon} \right\|_1 \leq C(\|\epsilon\tilde{F}\|_1 + \|\tilde{\mathbf{v}} \cdot \nabla\tilde{\mathbf{v}}\|_1 + \|\tilde{\theta}\nabla\eta\|_1 + \|\eta\nabla\tilde{\theta}\|_1 + \|U \cdot \nabla\mathbf{v}\|_1 + \|\operatorname{div}\mathbf{v}\|_2),$$

which together with (2.25), (2.28) and (2.29) yields

$$\begin{aligned} \|\mathbf{v}\|_3 + \left\| \frac{\nabla\eta + \nabla\theta}{\epsilon} \right\|_1 &\leq C_6(1 + \|U\|_2^2) \left\{ \epsilon(\|\tilde{F}\|_1 + \|\tilde{G}\|_{-1}) + \|\tilde{\mathbf{v}}\|_3^2 + \|\tilde{\theta}\|_3\|\eta\|_2 \right. \\ &\quad \left. + \|\tilde{\mathbf{v}}\|_3^{1/2}\|\eta\|_0 + \left[\epsilon\|P\|_2(\|U\|_2 + \|\tilde{\mathbf{v}}\|_2)\|\eta\|_0 \right]^{1/2} + \|\tilde{\mathbf{v}}\|_1(\|\tilde{\theta}\|_1 + \|\eta\|_1) \right\} + \|\nabla^2\operatorname{div}\mathbf{v}\|_0, \end{aligned} \quad (2.30)$$

where C_6 is a positive constant.

2.2.3 Estimate of $\|\nabla^2\operatorname{div}\mathbf{v}\|_0$

As in [23, 25, 12], in order to control the term $\|\nabla^2\operatorname{div}\mathbf{v}\|_0$ we divide it into the interior part and the part near the boundary. We remark that here we have to carefully deal with the terms which involve with the large parameter $1/\epsilon$ in (2.10).

I. Interior estimate

First, we derive the interior estimate of $\nabla^2\operatorname{div}\mathbf{v}$ by using the estimate (2.29). Let χ_0 be a C_0^∞ -function, then we have

Lemma 2.5. *There is a positive constant C_7 independent of ϵ , such that*

$$\begin{aligned} &\mu\|\chi_0\nabla^2\mathbf{v}\|_0^2 + \zeta\|\chi_0\nabla\operatorname{div}\mathbf{v}\|_0^2 + \kappa\|\chi_0\nabla^2\theta\|_0^2 \\ &\leq C_7 \left[(\|U\|_3 + \|\tilde{\mathbf{v}}\|_3)\|\eta\|_1^2 + \epsilon^2(\|\tilde{F}\|_0^2 + \|\tilde{G}\|_0^2) + \epsilon\|P\|_2(\|U\|_2 + \|\tilde{\mathbf{v}}\|_2)\|\eta\|_1 + \|\tilde{\mathbf{v}}\|_2^4 \right. \\ &\quad \left. + \|\eta\|_2^2\|\tilde{\theta}\|_2^2 + \|U\|_3(\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) + \|\tilde{\mathbf{v}}\|_2^2(\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2) + \|\mathbf{v}\|_1^2 \right] + \delta \left\| \frac{\nabla\eta + \nabla\theta}{\epsilon} \right\|_0^2. \end{aligned} \quad (2.31)$$

Proof. We differentiate (2.10) with respect to x to get that

$$\begin{cases} U^j\partial_{ij}^2\eta + \frac{\partial_i\operatorname{div}\mathbf{v}}{\epsilon} = -\tilde{v}^j\partial_{ij}^2\eta - \partial_i(U^j + \tilde{v}^j)\partial_j\eta - \tilde{v}^j\partial_{ij}^2\eta - \partial_i(\eta\operatorname{div}\tilde{\mathbf{v}}) - \epsilon\partial_i\operatorname{div}(P(U + \tilde{\mathbf{v}})), \\ U^j\partial_{ij}^2v^k + \partial_iU^j\partial_jv^k - \mu\partial_{ijj}^3v^k - \zeta\partial_{ik}^2\operatorname{div}\mathbf{v} + \frac{\partial_{ik}^2\eta + \partial_{ik}^2\theta}{\epsilon} = \epsilon\partial_i\tilde{F}^k - \partial_i(\tilde{v}^j\partial_jv^k) - \partial_{ik}^2(\tilde{\theta}\eta), \\ U^j\partial_{ij}^2\theta + \partial_iU^j\partial_j\theta - \kappa\partial_{ijj}^3\theta + \frac{\partial_i\operatorname{div}\mathbf{v}}{\epsilon} = \epsilon\partial_i\tilde{G} - \partial_i(\tilde{v}^j\partial_j\tilde{\theta}) - \partial_i((\eta + \tilde{\theta})\operatorname{div}\tilde{\mathbf{v}}). \end{cases} \quad (2.32)$$

Multiplying (2.32)₁, (2.32)₂ and (2.32)₃ by $\chi_0^2\partial_i\eta$, $\chi_0^2\partial_iv^k$ and $\chi_0^2\partial_i\theta$ in L^2 respectively, and summing up the resulting equations, we find that

$$\begin{aligned} &\mu\|\chi_0\partial_{ij}^2v^k\|_0^2 + \zeta\|\chi_0\partial_i\operatorname{div}\mathbf{v}\|_0^2 + \kappa\|\chi_0\partial_{ij}^2\theta\|_0^2 \\ &= -\frac{1}{\epsilon} \int \chi_0^2\partial_i\operatorname{div}\mathbf{v}(\partial_i\eta + \partial_i\theta) + \chi_0^2\partial_iv^k\partial_k(\partial_i\eta + \partial_i\theta)dx \\ &\quad - \int (2\mu\chi_0\partial_j\chi_0\partial_{ij}^2v^k\partial_iv^k + 2\zeta\chi_0\partial_k\chi_0\partial_i\operatorname{div}\mathbf{v}\partial_iv^k + 2\kappa\chi_0\partial_j\chi_0\partial_{ij}^2\theta\partial_i\theta)dx \\ &\quad - \int \chi_0^2 \left[\partial_i(U^j + \tilde{v}^j)\partial_j\eta + (U^j + \tilde{v}^j)\partial_{ji}^2\eta + \partial_i\eta\operatorname{div}\tilde{\mathbf{v}} + \eta\partial_i\operatorname{div}\tilde{\mathbf{v}} + \epsilon\partial_i\operatorname{div}(P(U + \tilde{\mathbf{v}})) \right] \partial_i\eta dx \\ &\quad + \int \chi_0^2 (\epsilon\partial_i\tilde{F}^k - \partial_i(\tilde{\mathbf{v}} \cdot \nabla v^k) - \partial_{ik}^2(\tilde{\theta}\eta)) \partial_iv^k dx \\ &\quad + \int \chi_0^2 (\epsilon\partial_i\tilde{G} - \partial_i(\tilde{v}^j\partial_j\tilde{\theta}) - \partial_i((\eta + \tilde{\theta})\operatorname{div}\tilde{\mathbf{v}})) \partial_i\theta dx. \end{aligned}$$

If we apply partial integrations to the above identity, employ Sobolev's and Young's inequalities and the fact that

$$\begin{aligned}
& -\frac{1}{\epsilon} \int \chi_0^2 \partial_i \operatorname{div} \mathbf{v} (\partial_i \eta + \partial_i \theta) + \chi_0^2 \partial_i v^k \partial_k (\partial_i \eta + \partial_i \theta) dx \\
& = \frac{1}{\epsilon} \int 2\chi_0 \partial_k \chi_0 \partial_i v^k (\partial_i \eta + \partial_i \theta) dx + \frac{1}{\epsilon} \int \chi_0^2 \partial_i v^k \partial_k (\partial_i \eta + \partial_i \theta) - \chi_0^2 \partial_i v^k \partial_k (\partial_i \eta + \partial_i \theta) dx \\
& = \frac{1}{\epsilon} \int 2\chi_0 \partial_k \chi_0 \partial_i v^k (\partial_i \eta + \partial_i \theta) dx \leq \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0^2 + C_\delta \|\mathbf{v}\|_1^2,
\end{aligned}$$

we infer by summing up i, j, k that

$$\begin{aligned}
& \mu \|\chi_0 \nabla^2 \mathbf{v}\|_0^2 + \zeta \|\chi_0 \nabla \operatorname{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \nabla^2 \theta\|_0^2 \\
& \leq (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_1^2 + \delta \left(\|\mathbf{v}\|_2^2 + \|\theta\|_2^2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0^2 \right) \\
& \quad + C_\delta \left[\epsilon^2 (\|\tilde{F}\|_0^2 + \|\tilde{G}\|_0^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_1 + \|\tilde{\mathbf{v}}\|_2^4 + \|\eta\|_2^2 \|\tilde{\theta}\|_2^2 \right. \\
& \quad \left. + \|U\|_3 (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) + \|\tilde{\mathbf{v}}\|_2^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2) + \|\mathbf{v}\|_1^2 \right],
\end{aligned}$$

which, by using Poincaré's inequality and choosing δ appropriately small, implies the lemma. \square

Lemma 2.6. *There is a positive constant C_8 independent of ϵ , such that*

$$\begin{aligned}
& \mu \|\chi_0 \nabla^3 \mathbf{v}\|_0^2 + \zeta \|\chi_0 \nabla^2 \operatorname{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \nabla^3 \theta\|_0^2 \\
& \leq C_8 \left[(\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2^2 + \epsilon^2 (\|\tilde{F}\|_1^2 + \|\tilde{G}\|_1^2) + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2 + \|\tilde{\mathbf{v}}\|_3^4 \right. \\
& \quad \left. + \|\eta\|_2^2 \|\tilde{\theta}\|_2^2 + \|U\|_3 (\|\mathbf{v}\|_2^2 + \|\theta\|_2^2) + \|\tilde{\mathbf{v}}\|_2^2 (\|\tilde{\theta}\|_2^2 + \|\eta\|_2^2) + \|\mathbf{v}\|_2^2 \right] + \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2.
\end{aligned} \tag{2.32}$$

Proof. We differentiate (2.10) twice with respect to x to get that

$$\left\{ \begin{aligned}
& U^j \partial_{ilj}^3 \eta + \frac{\partial_i \operatorname{div} \mathbf{v}}{\epsilon} = -\tilde{v}^j \partial_{ilj}^3 \eta - \partial_{il}^2 (U^j + \tilde{v}^j) \partial_j \eta - \partial_i (U^j + \tilde{v}^j) \partial_{jl}^2 \eta \\
& \quad - \partial_l (U^j + \tilde{v}^j) \partial_{ij}^2 \eta - \partial_{il} (\eta \operatorname{div} \tilde{\mathbf{v}}) - \epsilon \partial_{il}^2 \operatorname{div} (P(U + \tilde{\mathbf{v}})), \\
& U^j \partial_{ijl}^3 v^k + \partial_l U^j \partial_{ij}^2 v^k + \partial_{il}^2 U^j \partial_j v^k + \partial_i U^j \partial_{lj}^2 v^k - \mu \partial_{iljj}^4 v^k - \zeta \partial_{ilk}^3 \operatorname{div} \mathbf{v} + \frac{\partial_{ilk}^3 \eta + \partial_{ilk}^3 \theta}{\epsilon} \\
& \quad = \epsilon \partial_{il}^2 \tilde{F}^k - \partial_{il}^2 (\tilde{v}^j \partial_j \tilde{v}^k) - \partial_{ilk}^3 (\tilde{\theta} \eta), \\
& U^j \partial_{ijl}^3 \theta + \partial_l U^j \partial_{ij}^2 \theta + \partial_{il}^2 U^j \partial_j \theta + \partial_i U^j \partial_{lj}^2 \theta - \kappa \partial_{iljj}^4 \theta + \frac{\partial_{il}^2 \operatorname{div} \mathbf{v}}{\epsilon} \\
& \quad = \epsilon \partial_{il}^2 \tilde{G} - \partial_{il}^2 (\tilde{v}^j \partial_j \tilde{\theta}) - \partial_{il}^2 ((\eta + \tilde{\theta}) \operatorname{div} \tilde{\mathbf{v}}).
\end{aligned} \right. \tag{2.33}$$

Multiplying (2.33)₁, (2.33)₂ and (2.33)₃ again by $\chi_0^2 \partial_{il}^2 \eta$, $\chi_0^2 \partial_{il}^2 v^k$ and $\chi_0^2 \partial_{il}^2 \theta$ respectively, and summing up the resulting equations, we deduce that

$$\begin{aligned}
& \mu \|\chi_0 \partial_{ilj}^3 v^k\|_0^2 + \zeta \|\chi_0 \partial_{il}^2 \operatorname{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \partial_{ilj}^3 \theta\|_0^2 \\
& = -\frac{1}{\epsilon} \int \chi_0^2 \partial_{il} \operatorname{div} \mathbf{v} (\partial_{il} \eta + \partial_{il}^2 \theta) + \chi_0^2 \partial_{il}^2 v^k \partial_k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) dx \\
& \quad - \int 2\chi_0 \partial_j \chi_0 (\mu \partial_{ilj}^3 v^k \partial_{il}^2 v^k + \kappa \partial_{ilj}^3 \theta \partial_{il}^2 \theta) + 2\zeta \chi_0 \partial_k \chi_0 \partial_{il} \operatorname{div} v \partial_{il} v^k dx \\
& \quad - \int \left[U^j \partial_{ilj}^3 \eta + \tilde{v}^j \partial_{ilj}^3 \eta + \partial_{il}^2 (U^j + \tilde{v}^j) \partial_j \eta + \partial_i (U^j + \tilde{v}^j) \partial_{jl}^2 \eta \right. \\
& \quad \left. + \partial_l (U^j + \tilde{v}^j) \partial_{ij}^2 \eta + \partial_{il}^2 (\eta \operatorname{div} \tilde{\mathbf{v}}) + \epsilon \partial_{il}^2 \operatorname{div} (P(U + \tilde{\mathbf{v}})) \right] \chi_0^2 \partial_{il}^2 \eta dx
\end{aligned}$$

$$\begin{aligned}
& + \int \left[\epsilon \partial_{il}^2 \tilde{F}^k - \partial_{il}^2 (\tilde{v}^j \partial_j \tilde{v}^k) - \partial_{ilk}^3 (\tilde{\theta} \eta) - (U^j \partial_{ijl}^3 v^k + \partial_l U^j \partial_{ij}^2 v^k \right. \\
& \quad \left. + \partial_{il}^2 U^j \partial_j v^k + \partial_i U^j \partial_{lj}^2 v^k) \right] \chi_0^2 \partial_{il}^2 v^k dx \\
& + \int \left[\epsilon \partial_{il}^2 \tilde{G} - \partial_{il}^2 (\tilde{v}^j \partial_j \tilde{\theta}) - \partial_{il}^2 ((\eta + \tilde{\theta}) \operatorname{div} \tilde{\mathbf{v}}) - (U^j \partial_{ijl}^3 \theta + \partial_l U^j \partial_{ij}^2 \theta \right. \\
& \quad \left. + \partial_{il}^2 U^j \partial_j \theta + \partial_i U^j \partial_{lj}^2 \theta) \right] \chi_0^2 \partial_{il} \theta dx.
\end{aligned}$$

We integrate by parts the above identity, utilize Sobolev's inequality and the fact that

$$\begin{aligned}
& -\frac{1}{\epsilon} \int \chi_0^2 \partial_{il}^2 \operatorname{div} \mathbf{v} (\partial_{il}^2 \eta + \partial_{il}^2 \theta) + \chi_0^2 \partial_{il}^2 v^k \partial_k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) dx \\
& = \frac{1}{\epsilon} \int \chi_0^2 \partial_{il}^2 v^k \partial_k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) - \chi_0^2 \partial_{il}^2 v^k \partial_k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) dx + \frac{1}{\epsilon} \int 2 \chi_0 \partial_k \chi_0 \partial_{il}^2 v^k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) dx \\
& = \frac{1}{\epsilon} \int 2 \chi_0 \partial_k \chi_0 \partial_{il}^2 v^k (\partial_{il}^2 \eta + \partial_{il}^2 \theta) dx \leq \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2 + C_\delta \|\mathbf{v}\|_2^2,
\end{aligned}$$

and sum up i, j, k to infer that

$$\begin{aligned}
& \mu \|\chi_0 \nabla^3 \mathbf{v}\|_0^2 + \zeta \|\chi_0 \nabla^2 \operatorname{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \nabla^3 \theta\|_0^2 \\
& \leq (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2^2 + \delta (\|\mathbf{v}\|_3^2 + \|\theta\|_3^2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2) \\
& \quad + C_\delta [\epsilon^2 (\|\tilde{F}\|_1^2 + \|\tilde{G}\|_1^2) + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2 + \|\tilde{\mathbf{v}}\|_3^4 + \|\eta\|_2^2 \|\tilde{\theta}\|_2^2 \\
& \quad + \|U\|_3 (\|\mathbf{v}\|_2^2 + \|\theta\|_2^2) + \|\tilde{\mathbf{v}}\|_2^2 (\|\tilde{\theta}\|_2^2 + \|\eta\|_2^2) + \|\mathbf{v}\|_2^2],
\end{aligned}$$

which, by employing Poincaré's inequality and choosing δ suitably small, gives the lemma. \square

II. Boundary estimate

Next, we shall use the method of local coordinates to bound $\nabla^2 \operatorname{div} \mathbf{v}$ in the vicinity of the boundary (also see [23, 25, 12]). For completeness, we briefly describe the local coordinates as follows. First, one constructs the local coordinates by the isothermal coordinates $\lambda(\varphi, \phi)$ to derive an estimate near the boundary (see also [23, 25]), where

$$\lambda_\varphi \cdot \lambda_\varphi > 0, \quad \lambda_\phi \cdot \lambda_\phi > 0, \quad \lambda_\varphi \cdot \lambda_\phi = 0.$$

The boundary $\partial\Omega$ can be covered by a finite number of bounded open sets $W^k \subset \mathbb{R}^3$, $k = 1, 2, \dots, L$, such that for any $x \in W^k \cap \Omega$,

$$x = \Lambda^k(\varphi, \phi, r) \equiv \lambda^k(\varphi, \phi) + r \mathbf{n}(\lambda^k(\varphi, \phi)), \quad (2.34)$$

where $\lambda^k(\varphi, \phi)$ is the isothermal coordinates and \mathbf{n} is the unit outer normal to $\partial\Omega$.

Without confusion, we will omit the superscript k in each W^k in the following. We construct the orthonormal system corresponding to the local coordinates by

$$e_1 := \frac{\lambda_\varphi}{|\lambda_\varphi|}, \quad e_2 := \frac{\lambda_\phi}{|\lambda_\phi|}, \quad e_3 := e_1 \times e_2 \equiv \mathbf{n}(\lambda). \quad (2.35)$$

By a straightforward calculation, we see that for sufficiently small r and $J \in C^2$,

$$\begin{aligned}
J & := \det \operatorname{Jac} \Lambda = \det \frac{\partial x}{\partial(\varphi, \phi, r)} = \Lambda_\varphi \times \Lambda_\phi \cdot e_3 \\
& = |\lambda_\varphi| |\lambda_\phi| + r (|\lambda_\varphi| \mathbf{n}_\phi \cdot e_2 + |\lambda_\phi| \mathbf{n}_\varphi \cdot e_1) + r^2 [(\mathbf{n}_\varphi \cdot e_1)(\mathbf{n}_\phi \cdot e_2) - (\mathbf{n}_\varphi \cdot e_2)(\mathbf{n}_\phi \cdot e_1)] > 0.
\end{aligned}$$

And, we can easily derive the following relations as $(\text{Jac}\Lambda^{-1}) \circ \Lambda = (\text{Jac}\Lambda)^{-1}$ (also see [23]):

$$[\nabla(\Lambda^{-1})^1] \circ \Lambda = J^{-1}(\Lambda_\phi \times e_3), \quad (2.36)$$

$$[\nabla(\Lambda^{-1})^2] \circ \Lambda = J^{-1}(e_3 \times \Lambda_\varphi), \quad (2.37)$$

$$[\nabla(\Lambda^{-1})^3] \circ \Lambda = J^{-1}(\Lambda_\varphi \times \Lambda_\phi) = e_3, \quad (2.38)$$

where the symbol \circ stands for the composite of operators. Set $y := (\varphi, \phi, r)$, and denote by D_i the partial derivative with respect to y_i in local coordinates. We set the unknowns in local coordinates

$$\hat{\eta}(t, y) := \eta(t, \Lambda(y)), \quad \hat{\mathbf{v}}(t, y) := \mathbf{v}(t, \Lambda(y)), \quad \hat{\theta}(t, y) := \theta(t, \Lambda(y)),$$

and the knowns

$$\hat{U}(t, y) := U(t, \Lambda(y)), \quad \hat{\mathbf{v}}(t, y) := \hat{\mathbf{v}}(t, \Lambda(y)), \quad \hat{\theta}(t, y) := \tilde{\theta}(t, \Lambda(y)).$$

Then, we rewrite the system (2.10) in $[0, T] \times \tilde{\Omega}$, where $\tilde{\Omega} := \Lambda^{-1}(W \cap \Omega)$, as follows.

$$\begin{cases} \hat{U}^j a_{kj} D_k \hat{\eta} + \frac{a_{kj} D_k \hat{v}^j}{\epsilon} = -\hat{v}^j a_{kj} D_k \hat{\eta} - \hat{\eta} a_{kj} D_k \hat{v}^j - \epsilon a_{kj} D_k (\hat{P}(\hat{U}^j + \hat{v}^j)), \\ \hat{U}^j a_{kj} D_k \hat{v}^i - \mu a_{kj} D_k (a_{lj} D_l \hat{v}^i) - \zeta a_{ki} D_k (a_{lj} D_l \hat{v}^j) + \frac{a_{ki} D_k (\hat{\eta} + \hat{\theta})}{\epsilon} \\ \quad = \epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\theta} a_{ki} D_k \hat{\eta} - \hat{\eta} a_{ki} D_k \hat{\theta}, \\ \hat{U}^j a_{kj} D_k \hat{\theta} - \kappa a_{kj} D_k (a_{lj} D_l \hat{\theta}) + \frac{a_{kj} D_k \hat{v}^j}{\epsilon} = \epsilon \hat{G} - \hat{v}^j a_{kj} D_k \hat{\theta} - \hat{\eta} a_{kj} D_k \hat{v}^j - \hat{\theta} a_{kj} D_k \hat{v}^j, \end{cases} \quad (2.39)$$

with boundary conditions

$$\hat{\mathbf{v}}(t, y) = 0, \quad \hat{\theta}(t, y) = 0 \quad \text{on } \partial\tilde{\Omega}, \quad (2.40)$$

where a_{ij} is the (i, j) -th entry of the matrix $\text{Jac}(\Lambda^{-1}) = \frac{\partial y}{\partial x}$. Clearly, a_{ij} is a C^2 -function, and it follows from (2.36)–(2.38) that

$$\sum_{j=1}^3 a_{3j} a_{3j} = |\mathbf{n}|^2 = 1, \quad \sum_{j=1}^3 a_{1j} a_{3j} = \sum_{j=1}^3 a_{2j} a_{3j} = 0. \quad (2.41)$$

Moreover, this localized system has the following properties (see also [23]):

Proposition 2.1. *$D_i(Ja_{ij}) = 0$, for $j = 1, 2, 3$; $\varsigma D_\tau \hat{\mathbf{v}} = 0$, $\varsigma D_\tau D_\xi \hat{\mathbf{v}} = 0$ on $\partial\tilde{\Omega}$ in the tangential directions $\tau, \xi = 1, 2$, where $\varsigma \in C_0^\infty(\Lambda^{-1}(W))$. Similarly, $\varsigma D_\tau \hat{\theta} = 0$, $\varsigma D_\tau D_\xi \hat{\theta} = 0$ on $\partial\tilde{\Omega}$.*

Recalling $D_j = \sum_{i=1}^3 a_{ji} \partial_i$, we will frequently make use of the following relations without pointing out explicitly in subsequent calculations:

$$\|D_y \hat{\mathbf{v}}\|_{L^p(\Omega)} \leq C \|\nabla_x v\|_{L^p(\Omega)}, \quad \|D_y^2 \hat{\mathbf{v}}\|_{L^p(\Omega)} \leq C \|\nabla_x v\|_{W^{1,p}(\Omega)}, \quad 1 \leq p \leq \infty. \quad (2.42)$$

The above inequalities apply to η , θ and U , \tilde{v} , $\tilde{\theta}$, too.

By virtue of the interpolation $\|\cdot\|_{H^2}^2 \leq \delta \|\cdot\|_{H^3}^2 + C_\delta \|\cdot\|_{H^1}^2$, the boundary estimate of $\|\nabla^2 \text{div} u\|_{L_t^2(L^2)}$ can be reduced to the boundedness of

$$\int_0^t \int_{\tilde{\Omega}} J \chi^2 |D_y^2(a_{ji} D_j U^i)| dy ds,$$

where χ is a $C_0^\infty(\Lambda^{-1}(W))$ -function. So, we can split the estimate of derivatives on the boundary into two parts: the estimate of derivatives in the tangential directions and in the normal

direction.

Part 1. Estimate of derivatives in the tangential directions

First, we apply $D_{\tau\xi}^2$ to (2.39) with τ, ξ being the tangential directions to $\partial\tilde{\Omega}$ to get

$$\left\{ \begin{aligned} & \hat{U}^j a_{kj} D_{k\tau\xi}^3 \hat{\eta} + \frac{1}{\epsilon} D_{\tau\xi}^2 [a_{kj} D_k \hat{v}^j] \\ &= D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\eta} + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{\eta} + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{\eta} \\ &\quad - D_{\tau\xi}^2 [\hat{v}^j a_{kj} D_k \hat{\eta} + \hat{\eta} a_{kj} D_k \hat{v}^j + \epsilon a_{kj} D_k (\hat{P}(\hat{U}^j + \hat{v}^j))], \\ & \hat{U}^j a_{kj} D_{k\tau\xi}^3 \hat{v}^i - D_{\tau\xi}^2 [\mu a_{kj} D_k (a_{lj} D_l \hat{v}^i) + \zeta a_{ki} D_k (a_{lj} D_l \hat{v}^j)] + \frac{1}{\epsilon} D_{\tau\xi}^2 [a_{ki} D_k (\hat{\eta} + \hat{\theta})] \\ &= D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{v}^i + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{v}^i + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{v}^i \\ &\quad + D_{\tau\xi}^2 [\epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\theta} a_{ki} D_k \hat{\eta} - \hat{\eta} a_{ki} D_k \hat{\theta}], \\ & \hat{U}^j a_{kj} D_{k\tau\xi}^3 \hat{\theta} - \kappa D_{\tau\xi}^2 [a_{kj} D_k (a_{lj} D_l \hat{\theta})] + \frac{1}{\epsilon} D_{\tau\xi}^2 [a_{kj} D_k \hat{v}^j] \\ &= D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\theta} + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{\theta} + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{\theta} \\ &\quad + D_{\tau\xi}^2 [\epsilon \hat{G} - \hat{v}^j a_{kj} D_k \hat{\theta} - \hat{\eta} a_{kj} D_k \hat{v}^j - \hat{\theta} a_{kj} D_k \hat{v}^j]. \end{aligned} \right. \quad (2.43)$$

We multiply (2.43)₁, (2.43)₂ and (2.43)₃ by $J\chi^2 D_{\tau\xi} \hat{\eta}$, $J\chi^2 D_{\tau\xi} \hat{v}^i$ and $J\chi^2 D_{\tau\xi} \hat{\theta}$ respectively, and integrate the resulting identities to deduce that

$$\begin{aligned} & - \int_{\tilde{\Omega}} D_{\tau\xi}^2 [\mu a_{kj} D_k (a_{lj} D_l \hat{v}^i) + \zeta a_{ki} D_k (a_{lj} D_l \hat{v}^j)] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy \\ & - \int_{\tilde{\Omega}} \kappa D_{\tau\xi}^2 [a_{kj} D_k (a_{lj} D_l \hat{\theta})] \cdot J\chi^2 D_{\tau\xi}^2 \hat{\theta} dy \\ & + \int_{\tilde{\Omega}} \hat{U}^j a_{kj} (D_{k\tau\xi}^3 \hat{\eta} \cdot J\chi^2 D_{\tau\xi}^2 \hat{\eta} + D_{k\tau\xi}^3 \hat{v}^i \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i + D_{k\tau\xi}^3 \hat{\theta} \cdot J\chi^2 D_{\tau\xi}^2 \hat{\theta}) dy \\ & + \frac{1}{\epsilon} \int_{\tilde{\Omega}} \{ D_{\tau\xi}^2 [a_{kj} D_k \hat{v}^j] \cdot J\chi^2 D_{\tau\xi}^2 \hat{\eta} + D_{\tau\xi}^2 [a_{ki} D_k (\hat{\eta} + \hat{\theta})] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i + D_{\tau\xi}^2 [a_{kj} D_k \hat{v}^j] \cdot J\chi^2 D_{\tau\xi}^2 \hat{\theta} \} dy \\ & = \int_{\tilde{\Omega}} \{ D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\eta} + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{\eta} + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{\eta} \\ & \quad - D_{\tau\xi}^2 [\hat{v}^j a_{kj} D_k \hat{\eta} + \hat{\eta} a_{kj} D_k \hat{v}^j + \epsilon a_{kj} D_k (\hat{P}(\hat{U}^j + \hat{v}^j))] \} \cdot J\chi^2 D_{\tau\xi}^2 \hat{\eta} dy \\ & + \int_{\tilde{\Omega}} \{ D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{v}^i + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{v}^i + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{v}^i \\ & \quad + D_{\tau\xi}^2 [\epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\theta} a_{ki} D_k \hat{\eta} - \hat{\eta} a_{ki} D_k \hat{\theta}] \} \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy \\ & + \int_{\tilde{\Omega}} \{ D_\xi (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\theta} + D_{\tau\xi}^2 (\hat{U}^j a_{kj}) D_k \hat{\theta} + D_\tau (\hat{U}^j a_{kj}) D_{k\xi}^2 \hat{\theta} \\ & \quad + D_{\tau\xi}^2 [\epsilon \hat{G} - \hat{v}^j a_{kj} D_k \hat{\theta} - \hat{\eta} a_{kj} D_k \hat{v}^j - \hat{\theta} a_{kj} D_k \hat{v}^j] \} \cdot J\chi^2 D_{\tau\xi}^2 \hat{\theta} dy. \end{aligned} \quad (2.44)$$

Now, we denote LHS of (2.44) $:= L'_1 + L'_2 + L'_3 + L'_4$ and have to deal with each term due to integration by part and the boundary conditions.

$$\begin{aligned} L'_1 = & - \int_{\tilde{\Omega}} \left\{ D_{\tau\xi}^2 (\mu a_{kj}) D_k (a_{lj} D_l \hat{v}^i) + D_\tau (\mu a_{kj}) D_{k\xi}^2 (a_{lj} D_l \hat{v}^i) \right. \\ & + D_\xi (\mu a_{kj}) D_k [D_\tau (a_{lj}) D_l \hat{v}^i + a_{lj} D_{l\tau}^2 \hat{v}^i] + \mu a_{kj} D_{k\xi} [D_\tau (a_{lj}) D_l \hat{v}^i + a_{lj} D_{l\tau}^2 \hat{v}^i] \\ & \left. + D_{\tau\xi}^2 (\zeta a_{ki}) D_k (a_{lj} D_l \hat{v}^j) + D_\tau (\zeta a_{ki}) D_{k\xi}^2 (a_{lj} D_l \hat{v}^j) \right\} \end{aligned}$$

$$\begin{aligned}
& +D_\xi(\zeta a_{ki})D_k[D_\tau(a_{lj})D_l\hat{v}^j + a_{lj}D_{l\tau}^2\hat{v}^j] \\
& +\zeta a_{ki}D_{k\xi}^2[D_\tau(a_{lj})D_l\hat{v}^j + a_{lj}D_{l\tau}\hat{v}^j]\Big\} \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy
\end{aligned}$$

where

$$\begin{aligned}
& - \int_{\tilde{\Omega}} [\mu a_{kj} D_{k\xi}^2(a_{lj} D_{l\tau}^2 \hat{v}^i) + \zeta a_{ki} D_{k\xi}^2(a_{lj} D_{l\tau}^2 \hat{v}^j)] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy \\
& = \int_{\tilde{\Omega}} \mu J\chi^2 a_{kj} D_{k\tau\xi}^3 \hat{v}^i a_{lj} D_{l\tau\xi}^3 \hat{v}^i + \zeta J\chi^2 a_{ki} D_{k\tau\xi}^3 \hat{v}^i a_{lj} D_{l\tau\xi}^3 \hat{v}^j dy \\
& + \int_{\tilde{\Omega}} [D_k(\mu J\chi^2 a_{kj}) D_\xi(a_{lj} D_{l\tau} \hat{v}^i) \cdot D_{\tau\xi}^2 \hat{v}^i + \mu J\chi^2 a_{kj} D_{k\tau\xi}^3 \hat{v}^i D_\xi(a_{lj}) D_{l\tau}^2 \hat{v}^i \\
& + D_k(\zeta J\chi^2 a_{ki}) D_\xi(a_{lj} D_{l\tau}^2 \hat{v}^i) \cdot D_{\tau\xi}^2 \hat{v}^i + \mu J\chi^2 a_{ki} D_{k\tau\xi}^3 \hat{v}^i D_\xi(a_{lj}) D_{l\tau}^2 \hat{v}^j] dy,
\end{aligned}$$

and

$$\begin{aligned}
L'_2 & = \int_{\tilde{\Omega}} \kappa J\chi^2 a_{kj} D_{k\tau\xi}^3 \hat{\theta} a_{lj} D_{l\tau\xi}^3 \hat{\theta} dy \\
& + \int_{\tilde{\Omega}} \left[D_k(\kappa J\chi^2 a_{kj}) D_\xi(a_{lj} D_{l\tau}^2 \hat{\theta}) D_{\tau\xi}^2 \hat{\theta} + \kappa J\chi^2 a_{kj} D_{k\tau\xi}^3 \hat{\theta} D_\xi(a_{lj}) D_{l\tau}^2 \hat{\theta} \right] dy \\
& - \int_{\tilde{\Omega}} \left\{ D_{\tau\xi}^2(\kappa a_{kj}) D_k(a_{lj} D_l \hat{\theta}) + D_\tau(\kappa a_{kj}) D_{k\xi}^2(a_{lj} D_l \hat{\theta}) \right. \\
& \left. + D_\xi(\kappa a_{kj}) D_k \left[D_\tau(a_{lj}) D_l \hat{\theta} + a_{lj} D_{l\tau}^2 \hat{\theta} \right] + \kappa a_{kj} D_{k\xi}^2 D_\tau(a_{lj}) D_l \hat{\theta} \right\} \cdot J\chi^2 D_{\tau\xi}^2 \hat{\theta} dy.
\end{aligned}$$

On the other hand, recalling that $a_{kj} D_k \hat{U}^j = 0$, we have

$$\begin{aligned}
L'_3 & = -\frac{1}{2} \int_{\tilde{\Omega}} J\chi^2 a_{kj} D_k \hat{U}^j (|D_{\tau\xi}^2 \hat{\eta}|^2 + |D_{\tau\xi}^2 \hat{v}^i|^2 + |D_{\tau\xi}^2 \hat{\theta}|^2) dy \\
& - \frac{1}{2} \int_{\tilde{\Omega}} D_k(J\chi^2 a_{kj}) \hat{U}^j (|D_{\tau\xi}^2 \hat{\eta}|^2 + |D_{\tau\xi}^2 \hat{v}^i|^2 + |D_{\tau\xi}^2 \hat{\theta}|^2) dy \\
& = -\frac{1}{2} \int_{\tilde{\Omega}} D_k(J\chi^2 a_{kj}) \hat{U}^j (|D_{\tau\xi}^2 \hat{\eta}|^2 + |D_{\tau\xi}^2 \hat{v}^i|^2 + |D_{\tau\xi}^2 \hat{\theta}|^2) dy.
\end{aligned}$$

As for L'_4 , in view of the following identity

$$\begin{aligned}
& \frac{1}{\epsilon} \int_{\tilde{\Omega}} D_{\tau\xi}^2[a_{ki} D_k(\hat{\eta} + \hat{\theta})] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy = -\frac{1}{\epsilon} \int_{\tilde{\Omega}} J\chi^2 D_{\tau\xi}^2(a_{ki} D_k \hat{v}^i) D_{\tau\xi}^2(\hat{\eta} + \hat{\theta}) dy \\
& + \frac{1}{\epsilon} \int_{\tilde{\Omega}} \left[D_{\tau\xi}^2(a_{ki}) D_k(\hat{\eta} + \hat{\theta}) + D_\tau(a_{ki}) D_{k\xi}^2(\hat{\eta} + \hat{\theta}) + D_\xi(a_{ki}) D_{k\tau}^2(\hat{\eta} + \hat{\theta}) \right] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy \\
& + \frac{1}{\epsilon} \int_{\tilde{\Omega}} D_{\xi\tau}^2(\hat{\eta} + \hat{\theta}) [D_{\xi\tau}^2(a_{ki}) D_k \hat{v}^i + D_\tau(a_{ki}) D_{k\xi}^2 \hat{v}^i + D_\xi(a_{ki}) D_{k\tau}^2 \hat{v}^i \\
& - D_k(J\chi^2 a_{ki}) D_{\tau\xi}^2 \hat{v}^i - D_k(a_{ki}) J\chi^2 D_{\tau\xi}^2 \hat{v}^i] dy,
\end{aligned}$$

we deduce that

$$\begin{aligned}
L'_4 & = \frac{1}{\epsilon} \int_{\tilde{\Omega}} \left[D_{\tau\xi}^2(a_{ki}) D_k(\hat{\eta} + \hat{\theta}) + D_\tau(a_{ki}) D_{k\xi}^2(\hat{\eta} + \hat{\theta}) + D_\xi(a_{ki}) D_{k\tau}^2(\hat{\eta} + \hat{\theta}) \right] \cdot J\chi^2 D_{\tau\xi}^2 \hat{v}^i dy \\
& + \frac{1}{\epsilon} \int_{\tilde{\Omega}} D_{\xi\tau}^2(\hat{\eta} + \hat{\theta}) [D_{\xi\tau}^2(a_{ki}) D_k \hat{v}^i + D_\tau(a_{ki}) D_{k\xi}^2 \hat{v}^i + D_\xi(a_{ki}) D_{k\tau}^2 \hat{v}^i \\
& - D_k(J\chi^2 a_{ki}) D_{\tau\xi}^2 \hat{v}^i - D_k(a_{ki}) J\chi^2 D_{\tau\xi}^2 \hat{v}^i] dy.
\end{aligned}$$

Substituting the above estimates into (2.44), using Sobolev's and Young's inequalities and taking into account the property (2.42), we deduce that

$$\begin{aligned}
& \int_{\tilde{\Omega}} \mu J \chi^2 a_{kj} D_{k\tau\xi}^3 \hat{v}^i a_{lj} D_{l\tau\xi}^3 \hat{v}^i dy + \int_{\tilde{\Omega}} \zeta J \chi^2 a_{ki} D_{k\tau\xi}^3 \hat{v}^i a_{lj} D_{l\tau\xi}^3 \hat{v}^j dy + \int_{\tilde{\Omega}} \kappa J \chi^2 a_{kj} D_{k\tau\xi}^3 \hat{\theta} a_{lj} D_{l\tau\xi}^3 \hat{\theta} dy \\
& \leq C_9 \left[\|U\|_3 \|\eta\|_2^2 + \|\tilde{\mathbf{v}}\|_3 \|\eta\|_2^2 + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2 + \|U\|_3^2 (\|\mathbf{v}\|_1 + \|\theta\|_1) \right. \\
& \quad + \epsilon^2 (\|\tilde{F}\|_1^2 + \|\tilde{G}\|_1^2) + \|\tilde{\mathbf{v}}\|_2^4 + \|\eta\|_2^2 (\|\tilde{\mathbf{v}}\|_2^2 + \|\tilde{\theta}\|_2^2) + \|\tilde{\mathbf{v}}\|_2^2 \|\tilde{\theta}\|_2^2 \\
& \quad \left. + (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) \right] + \delta \left(\|\mathbf{v}\|_3^2 + \|\theta\|_3^2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2 \right), \tag{2.45}
\end{aligned}$$

where C_9 is a constant.

Part 2. Estimate of derivatives in the normal direction

We multiply (2.39)₂ by a_{3i} to obtain that

$$\begin{aligned}
& -(\mu + \zeta) D_3(a_{lj} D_l \hat{v}^j) + \frac{1}{\epsilon} D_3(\hat{\eta} + \hat{\theta}) \\
& = -a_{3i} \hat{U}^j a_{kj} D_k \hat{v}^i + \epsilon a_{3i} \hat{F}^i - a_{3i} \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\theta} D_3 \hat{\eta} - \hat{\eta} D_3 \hat{\theta} \\
& \quad + \mu \left[a_{3i} a_{kj} D_k (a_{lj} D_l \hat{v}^i) - D_3(a_{lj} D_l \hat{v}^j) \right], \tag{2.46}
\end{aligned}$$

where the last term in RHS of (2.46) can be written as follows.

$$\begin{aligned}
& \mu \left[a_{3i} a_{kj} D_k (a_{lj} D_l \hat{v}^i) - D_3(a_{lj} D_l \hat{v}^j) \right] \\
& = \mu \left[D_3(a_{3j}) D_3 \hat{v}^j + D_3(a_{\tau j}) D_\tau \hat{v}^j + a_{\tau j} D_{3\tau}^2 \hat{v}^j - a_{3j} D_3(a_{3j}) a_{3i} D_3 \hat{v}^i \right. \\
& \quad \left. - a_{\tau j} a_{3i} D_\tau a_{lj} D_l \hat{v}^i - a_{\tau j} a_{\xi j} a_{3i} D_{\tau\xi}^2 \hat{v}^i - a_{3j} a_{3i} D_3(a_{\tau j}) D_\tau \hat{v}^i \right], \quad \tau, \xi = 1, 2, \tag{2.47}
\end{aligned}$$

which does not include the term $D_{33} \hat{\mathbf{v}}$.

Step 1. To continue our estimate, we show the following lemma.

Lemma 2.7. *There are a constant C_{10} and a small $\delta > 0$, such that*

$$\begin{aligned}
& \frac{\mu + \zeta}{2} \int_{\tilde{\Omega}} J \chi^2 |D_{\tau 3}^2(a_{lj} D_l \hat{v}^j)|^2 dy + \frac{\kappa}{2} \int_{\tilde{\Omega}} J \chi^2 |D_{\tau 3}^2(a_{kj} D_k \hat{\theta})|^2 dy \\
& \leq C_{10} \left\{ \|U\|_4^2 \|\mathbf{v}\|_1^2 + \epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\eta\|_2^2 + \int_{\tilde{\Omega}} J \chi^2 |D_{\tau\xi y}^3 \mathbf{v}|^2 dy \right. \\
& \quad + \left[\|U\|_3 \|\eta\|_2^2 + \|\tilde{\mathbf{v}}\|_3 \|\eta\|_2^2 + (1 + \|U\|_3) \|\theta\|_2^2 + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2 \right. \\
& \quad \left. \left. + \epsilon^2 \|\tilde{G}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^2 \|\tilde{\theta}\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2 \|\eta\|_2^2 \right] \right\} + \delta (\|\mathbf{v}\|_3^2 + \|\theta\|_3^2). \tag{2.48}
\end{aligned}$$

Proof. We differentiate (2.46) with respect to y_τ ($\tau = 1, 2$), then multiply $-J \chi^2 D_{\tau 3} (a_{lj} D_l \hat{v}^j)$ in $L^2(\tilde{\Omega})$ to get

$$\begin{aligned}
& \frac{\mu + \zeta}{2} \int_{\tilde{\Omega}} J \chi^2 |D_{\tau 3}^2(a_{lj} D_l \hat{v}^j)|^2 dy - \frac{1}{\epsilon} \int_{\tilde{\Omega}} J \chi^2 D_{\tau 3}^2(\hat{\eta} + \hat{\theta}) D_{\tau 3}^2(a_{lj} D_l \hat{v}^j) dy \\
& \leq C (\|a_{3i} \hat{U}^j a_{kj} D_k \hat{v}^i\|_1^2 + \epsilon^2 \|\hat{F}^i\|_1^2 + \|a_{3i} \hat{v}^j a_{kj} D_k \hat{v}^i\|_1^2 \\
& \quad + \|\hat{\theta} D_3 \hat{\eta} + \hat{\eta} D_3 \hat{\theta}\|_1^2) + C \int_{\tilde{\Omega}} J \chi^2 |D_{\tau\xi y}^3 v|^2 dy \\
& \leq C (\|U\|_2^2 \|v\|_2^2 + \epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{v}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\hat{\eta}\|_2^2) + C \int_{\tilde{\Omega}} J \chi^2 |D_{\tau\xi y}^3 v|^2 dy. \tag{2.49}
\end{aligned}$$

In the mean while, we apply $D_{\tau 3}$ to $(2.39)_3$ and $(2.39)_1$, take the product of the resulting equations with $J\chi^2 D_{\tau 3}^2 \hat{\theta}$ and $J\chi^2 D_{\tau 3}^2 \hat{\eta}$ in $L^2(\tilde{\Omega})$, and sum then two identities to get

$$\begin{aligned}
& - \int_{\tilde{\Omega}} \kappa D_{\tau 3} \left[a_{kj} D_k (a_{lj} D_l \hat{\theta}) \right] \cdot J\chi^2 D_{\tau 3}^2 \hat{\theta} dy + \frac{1}{\epsilon} \int_{\tilde{\Omega}} D_{\tau 3}^2 (a_{kj} D_k \hat{v}^j) \cdot J\chi^2 (D_{\tau 3}^2 \hat{\theta} + D_{\tau 3}^2 \hat{\eta}) dy \\
& + \int_{\tilde{\Omega}} \left[D_{\tau 3}^2 (\hat{U}^j a_{kj} D_k \hat{\eta}) \cdot J\chi^2 D_{\tau 3}^2 \hat{\eta} + D_{\tau 3}^2 (\hat{U}^j a_{kj} D_k \hat{\theta}) \cdot J\chi^2 D_{\tau 3}^2 \hat{\theta} \right] dy \\
& = - \int_{\tilde{\Omega}} D_{\tau 3}^2 \left\{ \hat{v}^j a_{kj} D_k \hat{\eta} + \hat{\eta} a_{kj} D_k \hat{v}^j + \epsilon a_{kj} D_k \left[\hat{P} (\hat{U}^j + \hat{v}^j) \right] \right\} \cdot J\chi^2 D_{\tau 3}^2 \hat{\eta} \\
& + \int_{\tilde{\Omega}} D_{\tau 3}^2 \left(\epsilon \hat{G} - \hat{v}^j a_{kj} D_k \hat{\theta} - \hat{\eta} a_{kj} D_k \hat{v}^j - \hat{\theta} a_{kj} D_k \hat{v}^j \right) \cdot J\chi^2 D_{\tau 3}^2 \hat{\theta} dy.
\end{aligned} \tag{2.50}$$

We denote LHS of (2.50) := $L_1'' + L_2'' + L_3''$. To control L_k'' , we integrate by part to deduce that

$$\begin{aligned}
L_1'' & = \kappa \int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}^2 (a_{kj} D_k \hat{\theta}) D_{\tau 3}^2 (a_{lj} D_l \hat{\theta}) dy \\
& - \int_{\tilde{\Omega}} \kappa J\chi^2 D_{\tau 3}^2 \hat{\theta} \left(D_{\tau 3}^2 (a_{kj}) D_k (a_{lj} D_l \hat{\theta}) + D_{\tau} (a_{kj}) D_{3k}^2 (a_{lj} D_l \hat{\theta}) + D_3 (a_{kj}) D_{k\tau}^2 (a_{lj} D_l \hat{\theta}) \right) dy \\
& + \kappa \int_{\tilde{\Omega}} D_k (J\chi^2 a_{kj}) D_{\tau 3}^2 (\hat{\theta}) D_{\tau 3}^2 (a_{lj} D_l \hat{\theta}) dy \\
& - \kappa \int_{\tilde{\Omega}} J\chi^2 \left(D_{\tau 3}^2 (a_{kj}) D_k \hat{\theta} + D_{\tau} (a_{kj}) D_{3k}^2 \hat{\theta} + D_3 (a_{kj}) D_{k\tau}^2 \hat{\theta} \right) \cdot D_{\tau 3}^2 (a_{lj} D_l \hat{\theta}) dy
\end{aligned}$$

and

$$\begin{aligned}
L_3'' & = \int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}^2 \hat{\eta} \cdot \left(D_{\tau 3}^2 (\hat{U}^j a_{kj}) D_k \hat{\eta} + D_{\tau} (\hat{U}^j a_{kj}) D_{k3}^2 \hat{\eta} + D_3 (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\eta} \right) dy \\
& - \frac{1}{2} \int_{\tilde{\Omega}} D_k (J\chi^2 a_{kj}) \hat{U}^j |D_{\tau 3}^2 \hat{\eta}|^2 + J\chi^2 a_{kj} D_k \hat{U}^j |D_{\tau 3}^2|^2 dy \\
& + \int_{\tilde{\Omega}} \left(D_{\tau 3}^2 (\hat{U}^j a_{kj}) D_k \hat{\theta} + D_{\tau} (\hat{U}^j a_{kj}) D_{k3}^2 \hat{\theta} + D_3 (\hat{U}^j a_{kj}) D_{k\tau}^2 \hat{\theta} \right) \cdot J\chi^2 D_{\tau 3}^2 \hat{\theta} dy \\
& - \frac{1}{2} \int_{\tilde{\Omega}} D_k (J\chi^2 a_{kj}) \hat{U}^j |D_{\tau 3}^2 \hat{\theta}|^2 + J\chi^2 a_{kj} D_k \hat{U}^j |D_{\tau 3}^2 \hat{\theta}|^2 dy.
\end{aligned}$$

Inserting the estimates for L_1'' and L_3'' into (2.50), we obtain

$$\begin{aligned}
& \frac{\kappa}{2} \int_{\tilde{\Omega}} J\chi^2 |D_{\tau 3}^2 (a_{kj} D_k \hat{\theta})|^2 dy + \frac{1}{\epsilon} \int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}^2 (\hat{\eta} + \hat{\theta}) D_{\tau 3}^2 (a_{lj} D_l \hat{v}^j) dy \\
& \leq \|\theta\|_2^2 + \delta \left(\|D_{3k}^2 (a_{lj} D_l \hat{\theta})\|_0^2 + \|D_{k\tau}^2 (a_{lj} D_l \hat{\theta})\|_0^2 + \|\theta\|_3^2 \right) \\
& + C \left[\|U\|_3 \|\eta\|_2^2 + \|U\|_3 \|\theta\|_2^2 + \|\tilde{v}\|_3 \|\eta\|_2^2 + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{v}\|_3) \|\eta\|_2 \right. \\
& \left. + \epsilon^2 \|\tilde{G}\|_1^2 + \|\tilde{v}\|_2^2 \|\hat{\theta}\|_2^2 + \|\tilde{v}\|_2^2 \|\eta\|_2^2 \right].
\end{aligned} \tag{2.51}$$

Thanks to Sobolev's and Young's inequalities, we take the sum of (2.49) and (2.51) to deduce the estimate (2.48). \square

Step 2. Now, it suffices to bound $\|D_{33}^2 (a_{ij} D_i \hat{v}^j)\|_0$ in order to close the estimate for $\text{div} \mathbf{v}$. We apply D_3 to (2.46) to find that

$$-(\mu + \zeta) D_{33}^2 (a_{lj} D_l \hat{v}^j) + \frac{1}{\epsilon} D_{33}^2 (\hat{\eta} + \hat{\theta})$$

$$\begin{aligned}
&= D_3(a_{3i})(-\hat{U}^j a_{kj} D_k \hat{v}^i + \epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i) - a_{3i} D_3(\hat{U}^j a_{kj} D_k \hat{v}^i - \epsilon \hat{F}^i + \hat{v}^j a_{kj} D_k \hat{v}^i) \\
&\quad - D_3(\hat{\theta} D_3 \hat{\eta} + \hat{\eta} D_3 \hat{\theta}) + O(1)(D_{33\tau}^3 \hat{v}^j + D_{3l}^2 \hat{v}^j + D_l \hat{v}^j).
\end{aligned} \tag{2.52}$$

Now, multiplying the above equality (2.52) by $-J\chi^2 D_{33}^2(a_{lj} D_l \hat{v}^j)$ in $L^2(\tilde{\Omega})$, one infers that

$$\begin{aligned}
&\frac{\mu + \zeta}{2} \int_{\tilde{\Omega}} J\chi^2 |D_{33}^2(a_{lj} D_l \hat{v}^j)|^2 dy - \frac{1}{\epsilon} \int_{\tilde{\Omega}} J\chi^2 D_{33}^2(a_{lj} D_l \hat{v}^j) \cdot D_{33}^2(\hat{\eta} + \hat{\theta}) dy \\
&\leq C(\|U\|_2^2 \|v\|_2^2 + \epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{v}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\tilde{\eta}\|_2^2) + \|v\|_2^2 + \|v\|_1^2 + \int_{\tilde{\Omega}} J\chi^2 |D_{33\tau}^3 \hat{\mathbf{v}}|^2 dy.
\end{aligned} \tag{2.53}$$

Correspondingly, applying D_{33}^2 to (2.39)₁ and (2.39)₃ and multiplying the resulting equations by $J\chi^2 D_{33}^2 \hat{\eta}$ and $J\chi^2 D_{33}^2 \hat{\eta}$ respectively, we get

$$\begin{aligned}
&\frac{\kappa}{2} \int_{\tilde{\Omega}} J\chi^2 |D_{33}^2(a_{kj} D_k \hat{\theta})|^2 dy + \frac{1}{\epsilon} \int_{\tilde{\Omega}} J\chi^2 D_{33}^2(a_{lj} D_l \hat{v}^j) \cdot D_{33}^2(\hat{\eta} + \hat{\theta}) dy \\
&\leq C\|\theta\|_2^2(1 + \|U\|_3) + \delta\|\theta\|_3^2 + \|U\|_3\|\eta\|_2^2 + \|\tilde{v}\|_3\|\eta\|_2^2 + \epsilon\|P\|_3(\|U\|_3 + \|\tilde{v}\|_3)\|\eta\|_2 \\
&\quad + \epsilon^2\|\tilde{G}\|_1^2 + \|\tilde{v}\|_2^2\|\tilde{\theta}\|_2^2 + \|\tilde{v}\|_2^2\|\eta\|_2^2.
\end{aligned} \tag{2.54}$$

Combining (2.53) with (2.54), we see that there are a constant C_{11} and a small δ , such that

$$\begin{aligned}
&\frac{\mu + \zeta}{2} \int_{\tilde{\Omega}} J\chi^2 |D_{33}^2(a_{lj} D_l \hat{v}^j)|^2 dy + \frac{\kappa}{2} \int_{\tilde{\Omega}} J\chi^2 |D_{33}^2(a_{kj} D_k \hat{\theta})|^2 dy \\
&\leq \delta(\|\theta\|_3^2 + \|\mathbf{v}\|_3^2) + C_{11} \left\{ (\|U\|_4^2 \|\mathbf{v}\|_1^2 + \epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\tilde{\eta}\|_2^2) \right. \\
&\quad + \left[\|\mathbf{v}\|_1^2 + \int_{\tilde{\Omega}} J\chi^2 |D_{33\tau}^3 \hat{\mathbf{v}}|^2 dy + \|\theta\|_1^2(1 + \|U\|_3) + \|U\|_3\|\eta\|_2^2 + \|\tilde{\mathbf{v}}\|_3\|\eta\|_2^2 \right. \\
&\quad \left. \left. + \epsilon\|P\|_3(\|U\|_3 + \|\tilde{\mathbf{v}}\|_3)\|\eta\|_2 + \epsilon^2\|\tilde{G}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^2\|\tilde{\theta}\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2\|\eta\|_2^2 \right] \right\}.
\end{aligned} \tag{2.55}$$

Step 3. To control the term $D_{33\tau}^3 \hat{\mathbf{v}}$ on RHS of (2.55), we introduce an auxiliary Stokes problem in the original coordinates in the region near the boundary:

$$\begin{cases} -\mu \Delta_x [(\chi D_\tau \hat{\mathbf{v}}) \circ \Lambda^{-1}] + \frac{1}{\epsilon} \nabla_x [(\chi D_\tau (\hat{\eta} + \hat{\theta})) \circ \Lambda^{-1}] = G_1 & \text{in } W \cap \Omega, \\ \operatorname{div}_x [(\chi D_\tau \hat{\mathbf{v}}) \circ \Lambda^{-1}] = G_2 & \text{in } W \cap \Omega, \\ (\chi D_\tau \hat{\mathbf{v}}) \circ \Lambda^{-1} = 0 & \text{in } W \cap \Omega, \end{cases}$$

where

$$\begin{aligned}
G_1^i &= \chi D_\tau \left[\zeta a_{ki} D_k (a_{lj} D_l \hat{v}^j) + \epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\theta} a_{ki} D_k \hat{\eta} - \hat{\eta} a_{ki} D_k \hat{\theta} \right. \\
&\quad \left. - \hat{U}^j a_{kj} D_k \hat{v}^i \right] + o(1) \left[D_l \hat{v}^i + D_{kl}^2 \hat{v}^i + \frac{1}{\epsilon} D_k (\hat{\eta} + \hat{\theta}) \right]
\end{aligned}$$

and

$$G_2^i = o(1)(D_\tau \hat{v}^j + D_k \hat{v}^j + D_{\tau k}^2 \hat{v}^j),$$

which can be bounded as follows.

$$\begin{aligned}
\|G_1\|_{L^2}^2 &\leq C \left(\left\| \frac{\nabla(\eta + \theta)}{\epsilon} \right\|_{L^2}^2 + \int_{\tilde{\Omega}} J\chi^2 |D_{\tau k}^2(a_{lj} D_l \hat{v}^j)|^2 dy \right) + \delta \|\mathbf{v}\|_3^2 \\
&\quad + C_\delta \|\mathbf{v}\|_1^2 + C(\epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\tilde{\eta}\|_2^2 + \|U\|_3^4 \|\mathbf{v}\|_1^2)
\end{aligned} \tag{2.56}$$

and

$$\|G_2\|_{H^1}^2 \leq \delta \|\mathbf{v}\|_3^2 + C_\delta \|\mathbf{v}\|_1^2 + C \int_{\tilde{\Omega}} J\chi^2 |D_{\tau k}(a_{lj} D_l \hat{v}^j)|^2 dy. \tag{2.57}$$

Due to the regularity theory of the Stokes problem (see [9]), one has

$$\int_{W \cap \Omega} |\Delta_x(\chi D_\tau \hat{\mathbf{v}}) \circ \Lambda^{-1}(x)|^2 dx \leq C(\|G_1\|_{L^2(W \cap \Omega)}^2 + \|G_2\|_{H^1(W \cap \Omega)}^2), \quad (2.58)$$

where the left-hand side of (2.58) is equal to

$$\begin{aligned} \int_{W \cap \Omega} |\Delta_x(\chi D_\tau \hat{\mathbf{v}}) \circ \Lambda^{-1}(x)|^2 dx &= \int_{\tilde{\Omega}} J \left| \sum_{j=1}^3 \sum_{k=1}^3 a_{kj} D_k \left(\sum_{l=1}^3 a_{lj} D_l(\chi D_\tau \hat{\mathbf{v}}) \right) \right|^2 dy \\ &= \int_{\tilde{\Omega}} J \chi^2 \left| \sum_{j,k,l=1}^3 a_{kj} a_{lj} D_{kl\tau}^3 \hat{\mathbf{v}} \right|^2 dy + o(1) \int_{\tilde{\Omega}} (|D_\tau \hat{\mathbf{v}}|^2 + |D_{y\tau}^2 \hat{\mathbf{v}}|^2) dy. \end{aligned}$$

And we use (2.41) to get

$$D_{33\tau}^3 \hat{\mathbf{v}} = \sum_{j,k,l=1}^3 a_{kj} a_{lj} D_{kl\tau}^3 \hat{\mathbf{v}} - \sum_{1 \leq k,l \leq 2} \sum_{j=1}^3 a_{kj} a_{lj} D_{kl\tau}^3 \hat{\mathbf{v}},$$

from which, (2.56) and (2.57) it follows that the inequality (2.58) gives

$$\begin{aligned} &(C_{11} + 1) \int_{\tilde{\Omega}} J \chi^2 |D_{33\tau}^3 \hat{\mathbf{v}}|^2 dy \\ &\leq C(\|G_1\|_{L^2(W \cap \Omega)}^2 + \|G_2\|_{H^1(W \cap \Omega)}^2) + C \int_{\tilde{\Omega}} J \chi^2 |D_{\tau\xi\zeta}^3|^2 dy + C_\delta \|\nabla \mathbf{v}\|_{L^2}^2 + \delta \|\mathbf{v}\|_3^2 \\ &\leq \delta \|\mathbf{v}\|_3^2 + C_{12} \left[\|\mathbf{v}\|_1^2 + \left\| \frac{\nabla(\eta + \theta)}{\epsilon} \right\|_0^2 + \int_{\tilde{\Omega}} J \chi^2 (|D_{\xi\tau y}^3 \hat{\mathbf{v}}|^2 + |D_{3\tau}^2(a_{lj} D_l \hat{\mathbf{v}}^j)|^2) dy \right. \\ &\quad \left. + (\epsilon^2 \|\tilde{F}\|_1^2 + \|\tilde{\mathbf{v}}\|_2^4 + \|\tilde{\theta}\|_2^2 \|\eta\|_2^2 + \|U\|_3^4 \|\mathbf{v}\|_1^2) \right]. \end{aligned} \quad (2.59)$$

Now, letting

$$\Phi_\chi := \int_{\tilde{\Omega}} J \chi^2 (|D_{\tau\xi y}^3 \hat{\mathbf{v}}|^2 + |D_{\tau 3}^2(a_{lj} D_l \hat{v}^j)|^2 + |D_{33}^2(a_{lj} D_l \hat{v}^j)|^2 + |D_{33\tau}^3 \hat{\mathbf{v}}|^2) dy$$

and

$$\Psi_\chi := \int_{\tilde{\Omega}} J \chi^2 (a_{kj} |D_{k\tau\xi}^3 \hat{\theta}|^2 + |D_{\tau 3}^2(a_{kj} D_k \hat{\theta})|^2 + |D_{33}^2(a_{kj} D_k \hat{\theta})|^2) dy,$$

we can apply Cauchy-Schwarz's and Young's inequalities as well as the estimate (2.29) to deduce from (2.45), (2.48), (2.55) and (2.59) that

$$\begin{aligned} \Phi_\chi + \Psi_\chi &\leq \|U\|_3 \|\eta\|_2^2 + (\|\tilde{\mathbf{v}}\|_3 + \|\tilde{\mathbf{v}}\|_2^2) \|\eta\|_2^2 + \epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) \|\eta\|_2 \\ &\quad + C(1 + \|U\|_3^4) (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) + \delta \left(\|\mathbf{v}\|_3^2 + \|\theta\|_3^2 + \left\| \frac{\nabla(\eta + \theta)}{\epsilon} \right\|_1^2 \right) \\ &\quad + C_\delta \left[\epsilon^2 (\|\tilde{F}\|_1^2 + \|\tilde{G}\|_1^2) + \|\tilde{\mathbf{v}}\|_2^4 + (\|\tilde{\theta}\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2) \|\eta\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2 \|\tilde{\theta}\|_2^2 \right], \end{aligned} \quad (2.60)$$

which, together with (2.30)–(2.32), results in

$$\begin{aligned} &\|\mathbf{v}\|_3^2 + \|\theta\|_3^2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2 \\ &\leq C_{12} \left\{ \|U\|_3 \|\eta\|_2^2 + (1 + \|U\|_3^4) \left[\epsilon \|P\|_3 (\|U\|_3 + \|\tilde{\mathbf{v}}\|_3) + (\|\tilde{\theta}\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2 + \|\tilde{\mathbf{v}}\|_3) \right] \|\eta\|_2^2 \right. \\ &\quad \left. + (1 + \|U\|_3^4) (\|\mathbf{v}\|_1^2 + \|\theta\|_1^2) + (1 + \|U\|_3^4) \left[\epsilon^2 (\|\tilde{F}\|_1^2 + \|\tilde{G}\|_1^2) + \|\tilde{\mathbf{v}}\|_2^4 + \|\tilde{\mathbf{v}}\|_2^2 \|\tilde{\theta}\|_2^2 \right] \right\}. \end{aligned} \quad (2.61)$$

2.2.4 Boundedness of η

In the next lemma, we derive upper bounds of $\|\eta\|_1$ and $\|\eta\|_2$.

Lemma 2.8. *There are a small $\delta > 0$, and two positive constants C_{13} and C_{14} independent of ϵ , such that*

$$\begin{aligned} \|\eta\|_1 \leq C_{13} & \left[\epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{\mathbf{v}}\|_2 \|\tilde{\mathbf{v}}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|\mathbf{v}\|_2 + \|\operatorname{div} \mathbf{v}\|_1 + (\|\tilde{F}\|_{-1} + \|\tilde{G}\|_{-1})) \right. \\ & \left. + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) + \|\tilde{\mathbf{v}}\|_3^{\frac{1}{2}} \|\eta\|_0 + \|\tilde{\mathbf{v}}\|_2^2 + \|\eta\|_1 \|\tilde{\theta}\|_1 + \|\tilde{\mathbf{v}}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right] \end{aligned} \quad (2.62)$$

and

$$\begin{aligned} \|\eta\|_2 \leq C_{14} (1 + \epsilon) (1 + \|U\|_2^2) & \left\{ \epsilon (\|\tilde{F}\|_1 + \|\tilde{G}\|_{-1}) + \|\tilde{\mathbf{v}}\|_3^2 + \|\tilde{\theta}\|_3 \|\eta\|_2 + \|\tilde{\mathbf{v}}\|_3^{\frac{1}{2}} \|\eta\|_0 \right. \\ & \left. + \left[\epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_0 \right]^{\frac{1}{2}} + \|\tilde{\mathbf{v}}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right\} + \epsilon \|\mathbf{v}\|_3 + \delta \|\theta\|_3. \end{aligned} \quad (2.63)$$

Proof. From (2.29) we get

$$\epsilon \|\mathbf{v}\|_2 + \|\nabla \eta + \nabla \theta\|_0 \leq C [\epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{\mathbf{v}}\|_2 \|\tilde{\mathbf{v}}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|\mathbf{v}\|_2 + \|\operatorname{div} \mathbf{v}\|_1)],$$

which together with Lemma 2.4 gives

$$\begin{aligned} \|\nabla \eta\|_1 & \leq \|\nabla \eta + \nabla \theta\|_0 + \|\nabla \theta\|_0 \\ & \leq C [\epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{\mathbf{v}}\|_2 \|\tilde{\mathbf{v}}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|\mathbf{v}\|_2 + \|\operatorname{div} \mathbf{v}\|_1)] \\ & \quad + C \left[\|\tilde{\mathbf{v}}\|_3 \|\eta\|_0^2 + \epsilon^2 (\|\tilde{F}\|_{-1}^2 + \|\tilde{G}\|_{-1}^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_0 \right. \\ & \quad \left. + \|\tilde{\mathbf{v}}\|_1^4 + \|\eta\|_1^2 \|\tilde{\theta}\|_1^2 + \|\tilde{\mathbf{v}}\|_1^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2) \right]^{1/2}. \end{aligned}$$

If we apply Poincaré's and Young's inequalities to the above inequality, and use the fact that

$$(A_1 + A_2 + \cdots + A_n)^{1/2} \leq A_1^{1/2} + A_2^{1/2} + \cdots + A_n^{1/2} \quad \text{for } A_i \geq 0 \quad (i = 1, \dots, n), \quad (2.64)$$

we obtain the estimate (2.62) immediately.

On the other hand, from the estimate (2.30) we conclude that

$$\begin{aligned} \|\nabla \eta\|_1 & \leq \|\nabla \eta + \nabla \theta\|_1 + \|\nabla \theta\|_1 \\ & \leq C \epsilon (1 + \|U\|_2^2) \left\{ \epsilon (\|\tilde{F}\|_1 + \|\tilde{G}\|_{-1}) + \|\tilde{\mathbf{v}}\|_3^2 + \|\tilde{\theta}\|_3 \|\eta\|_2 + \|\tilde{\mathbf{v}}\|_3^{\frac{1}{2}} \|\eta\|_0 \right. \\ & \quad \left. + \left[\epsilon \|P\|_2 (\|U\|_2 + \|\tilde{\mathbf{v}}\|_2) \|\eta\|_0 \right]^{\frac{1}{2}} + \|\tilde{\mathbf{v}}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right\} + \epsilon \|\mathbf{v}\|_3 + \delta \|\theta\|_3 + C_\delta \|\theta\|_1, \end{aligned}$$

which, together with Poincaré's inequality, (2.62) and (2.64), implies (2.63). \square

3 Existence of the nonlinear problem

In this section, we give the proof of the existence for the nonlinear problem (1.5) by using the Tikhonov Theorem which can be found in [20]. For completeness, we state the theorem in the following.

Theorem 3.1. *(Tikhonov Theorem, [20, P72, 1.2.11.6]) Let M be a nonempty bounded closed convex subset of a separable reflexive Banach space X and let $F : M \rightarrow M$ be a weakly continuous mapping (i.e., if $x_n \in M$, $x_n \rightharpoonup x$ weakly in X , then $F(x_n) \rightharpoonup F(x)$ weakly in X as well). Then F has at least one fixed point in M .*

Define a Banach space X by

$$X = \bar{H}^1 \times H_0^1 \times H_0^1,$$

which can be easily verified to be separable and reflexive.

A convex subset $K_1(E)$ of X is defined by

$$K_1(E) = \{(\mathbf{v}, \theta) \in (H^3 \cap H_0^1) \times (H^3 \cap H_0^1) \mid \|\mathbf{v}\|_3 + \|\theta\|_3 \leq E\},$$

where $E < 1$ is a small positive constant. By the lower semi-continuity of norms, we easily see that the subset $K_1(E)$ is also closed in X .

We define a space K by

$$K = K_0 \times K_1(E),$$

where K_0 is defined by (2.9). Note that K is a nonempty bounded closed convex subset of X .

Now, we define a nonlinear operator N from K to X by

$$N(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta}) := (U, \mathbf{v}, \theta),$$

where U and (\mathbf{v}, θ) are the solutions of (2.3) and (2.10) for given $(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta})$, respectively.

Next, we want to find a fixed point (U, \mathbf{v}, θ) of N in K , such that $(U, \mathbf{v}, \theta) = N(U, \mathbf{v}, \theta)$, which, together with the existence of weak solution in Lemmas 2.1 and 2.4, gives that (U, P) and $(\eta, \mathbf{v}, \theta)$ are solutions of the boundary value problems (2.1) and (2.2), respectively. So $(U + \mathbf{v}, \epsilon P + \eta, \theta)$ will be a solution to (1.5). For this purpose, we have to show that N maps K into itself and $N : K \rightarrow K$ is a weakly continuous mapping.

Lemma 3.1. *There is a small constant $\epsilon_0 > 0$, depending only on $\Omega, \mu, \lambda, \mathbf{f}$ and \mathbf{g} , such that for any $\epsilon \in (0, \epsilon_0)$, K is a nonempty bounded closed convex subset of X and $N(K) \subset K$.*

Proof. By virtue of the definition, it is obvious that $K \subset X$ is a nonempty, bounded, closed convex set. Now, we will show that the operator N maps K into itself, i.e., $N(K) \subset K$. To this end, let $(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta}) \in K$ and $(U, \mathbf{v}, \theta) = N(\tilde{U}, \tilde{\mathbf{v}}, \tilde{\theta})$. By Lemmas 2.1 and 2.2, we see that $U \in K_0$ for all $\tilde{U} \in K_0$. Thus, it suffices to check that $(\mathbf{v}, \theta) \in K_1(E)$ for $(\tilde{\mathbf{v}}, \tilde{\theta}) \in K_1(E)$. By (2.7) and (2.8), we have

$$\|U\|_3 + \|\nabla P\|_1 \leq M_1 \quad \text{and} \quad \|U\|_4 + \|\nabla P\|_2 \leq M_2, \quad (3.1)$$

where $M_1 = C_3 \|\mathbf{h}\|_1 (\|\mathbf{h}\|_1 + 1)^8$ and $M_2 = C_4 \|\mathbf{h}\|_2 (\|\mathbf{h}\|_2 + 1)^{12}$.

On the other hand, recalling the definition of \tilde{F} and \tilde{G} , we get from (3.1) that

$$\begin{aligned} \|\tilde{F}\|_1 &= \|(\epsilon P + \eta)\mathbf{f} - (\epsilon P + \eta)(U + \tilde{\mathbf{v}}) \cdot \nabla(U + \tilde{\mathbf{v}}) - \tilde{\theta} \nabla P - P \nabla \tilde{\theta}\|_1 \\ &\leq C[\|\eta\|_2(\|\mathbf{f}\|_1 + \|U\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2) + \epsilon\|P\|_2(\|\mathbf{f}\|_1 + \|U\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2 + \|\tilde{\theta}\|_2)] \\ &\leq C\|\eta\|_2(\|\mathbf{f}\|_1 + (M_1 + 1)^2) + \epsilon C M_1(\|\mathbf{f}\|_1 + (M_1 + 1)^2), \end{aligned} \quad (3.2)$$

$$\begin{aligned} \|\tilde{G}\|_1 &= \|\tilde{\Psi} - (\epsilon P + \eta)(U + \tilde{\mathbf{v}}) \cdot \nabla \tilde{\theta} + (\epsilon P + \eta) \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} + P \operatorname{div} \tilde{\mathbf{v}}\|_1 \\ &\leq C[\epsilon(\|U\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2) + (\|\eta\|_2 + \epsilon\|P\|_2)(\|U\|_2 + \|\tilde{\mathbf{v}}\|_2)\|\tilde{\theta}\|_2 + \|P\|_2\|\tilde{\mathbf{v}}\|_2] \\ &\leq C\|\eta\|_2(M_1 + 1) + C(\epsilon(M_1 + 1)^2 + (M_1 + 1)). \end{aligned} \quad (3.3)$$

As a result of Poincaré's inequality and Lemma 2.4, we have

$$\begin{aligned} \|\mathbf{v}\|_1 + \|\theta\|_1 &\leq C\left\{(E^{\frac{1}{2}} + E)\|\eta\|_2 + \epsilon\left[\|\eta\|_2(M_1 + 1) + C(\|\eta\|_2(\|\mathbf{f}\|_1 + (M_1 + 1)^2) \right. \right. \\ &\quad \left. \left. + \epsilon C M_1(\|\mathbf{f}\|_1 + (M_1 + 1)^2) + \epsilon(M_1 + 1)^2 + (M_1 + 1)\right]\right. \\ &\quad \left. + \epsilon C_\delta(M_1 + 1)^2 + E^2\right\} + \delta\|\eta\|_2 \end{aligned}$$

$$\begin{aligned} &\leq C[E^{\frac{1}{2}} + \epsilon(\|\mathbf{f}\|_1 + (M_1 + 1)^2)]\|\eta\|_2 + \epsilon^2 CM_1(\|\mathbf{f}\|_1 + (M_1 + 1)^2) + CE^2 \\ &\quad + \delta\|\eta\|_2 + \epsilon C_\delta(M_1 + 1)^2, \end{aligned} \quad (3.4)$$

where we have used the estimates (3.1), (3.2) and (3.3).

On the other hand, in view of Poincaré's and Young's inequalities, (3.4) and (2.63) in Lemma 2.8, we find that

$$\begin{aligned} \|\eta\|_2 &\leq C(1 + \epsilon)(1 + M_1^2) \left\{ \epsilon \left[\|\eta\|_2(\|\mathbf{f}\|_1 + (M_1 + 1)^2) + M_1(\|\mathbf{f}\|_1 + (M_1 + 1)^2) \right. \right. \\ &\quad \left. \left. + \|\eta\|_2(M_1 + 1) + (\epsilon(M_1 + 1)^2 + (M_1 + 1)) \right] + E^2 + (E + E^{\frac{1}{2}})\|\eta\|_2 \right. \\ &\quad \left. + \epsilon^{\frac{1}{2}} M_1^{\frac{1}{2}} (M_1 + E)^{\frac{1}{2}} \|\eta\|_2^{\frac{1}{2}} + E(E + \|\eta\|_2) \right\} + \epsilon\|\mathbf{v}\|_3 + \delta\|\theta\|_3 \\ &\leq C_{15}(1 + \epsilon)(1 + M_1^2) \left\{ \epsilon \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 + (M_1 + 1) \right] + (E + E^{\frac{1}{2}}) + \delta \right\} \|\eta\|_2 \\ &\quad + \epsilon\|\mathbf{v}\|_3 + \delta\|\theta\|_3 + C_{15}(1 + \epsilon)(1 + M_1^2) \left\{ \epsilon(M_1 + 1)^2 \right. \\ &\quad \left. + \epsilon \left[M_1(\|\mathbf{f}\|_1 + (M_1 + 1)^2) + (\epsilon(M_1 + 1)^2 + (M_1 + 1)) \right] + E^2 \right\}, \end{aligned} \quad (3.5)$$

where C_{15} is a positive constant.

Combining (2.61) with (3.2)–(3.5), we conclude that there is a constant C_{16} , such that

$$\begin{aligned} \|\mathbf{v}\|_3 + \|\theta\|_3 + \|\eta\|_2 &\leq C_{16}(1 + M_1)^5 \left\{ \epsilon \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) + \delta \right\} \|\eta\|_2 \\ &\quad + \epsilon M_1^{\frac{1}{2}} \|\mathbf{v}\|_3 + \delta M_1^{\frac{1}{2}} \|\theta\|_3 + C_{16}(1 + M_1)^5 \left\{ \epsilon(M_1 + 1)^2 \right. \\ &\quad \left. + \epsilon(1 + M_1) \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + E^2 \right\}. \end{aligned} \quad (3.6)$$

Thus, first taking δ small enough and then choosing ϵ_0 and E suitably small, such that

$$\begin{aligned} C_{16}(1 + M_1)^5 \left\{ \epsilon_0 \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) \right\} &< 1, \quad \epsilon_0 M_1^{\frac{1}{2}} < 1, \\ C_{16}(1 + M_1)^5 \left\{ \epsilon_0(M_1 + 1)^2 + \epsilon_0(1 + M_1) \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + E^2 \right\} &< E, \end{aligned}$$

we deduce from (3.6) that for all $\epsilon \in (0, \epsilon_0)$,

$$\|\mathbf{v}\|_3 + \|\theta\|_3 + \|\eta\|_2 \leq E,$$

which gives $\|\mathbf{v}\|_3 + \|\theta\|_3 \leq E$ immediately. This completes the proof. \square

Lemma 3.2. *Let N, X, K_0 and $K_1(E)$ be the same as in Lemma 3.1. Then $N : K \rightarrow K$ is a weakly continuous mapping.*

Proof. By the definition of weakly continuous mapping (see, for example, [20, P72,1.4.11.6]), it suffices to prove that N is continuous on K in the norm of X .

Let $(U_i, \mathbf{v}_i, \theta_i) = N(\tilde{U}_i, \tilde{\mathbf{v}}_i, \tilde{\theta}_i)$, $i = 1, 2$. In particular, let $(U_i, P_i) \in (H^4 \cap H_{0,\sigma}^1) \times \bar{H}^3$ and $(\eta_i, \mathbf{v}_i, \theta_i) \in \bar{H}^2 \times (H^3 \cap H_0^1) \times (H^3 \cap H_0^1)$ be the solutions of (2.3) and (2.10) for given $(\tilde{U}_i, \tilde{\mathbf{v}}_i, \tilde{\theta}_i)$ respectively, i.e.,

$$\begin{cases} (\tilde{U}_i + \tilde{\mathbf{v}}_i) \cdot \nabla U_i - \mu \Delta U_i + \nabla P_i = \mathbf{h}, & \int P_i dx = 0, \\ \operatorname{div} U_i = 0; \end{cases} \quad (3.7)$$

and

$$\begin{cases} U_i \cdot \nabla \eta_i + \frac{\operatorname{div} \mathbf{v}_i}{\epsilon} = -\tilde{\mathbf{v}}_i \cdot \nabla \eta_i - \eta_i \operatorname{div} \tilde{\mathbf{v}}_i - \epsilon \operatorname{div}(P_i(U_i + \tilde{\mathbf{v}}_i)), \\ U_i \cdot \nabla \mathbf{v}_i - \mu \Delta \mathbf{v}_i - \zeta \nabla \operatorname{div} \mathbf{v}_i + \frac{\nabla \eta_i + \nabla \theta_i}{\epsilon} = \epsilon \tilde{F}_i - \tilde{\mathbf{v}}_i \cdot \nabla \tilde{\mathbf{v}}_i - \tilde{\theta}_i \nabla \eta_i - \eta_i \nabla \tilde{\theta}_i, \\ U_i \cdot \nabla \theta_i - \kappa \Delta \theta_i + \frac{\operatorname{div} \mathbf{v}_i}{\epsilon} = \epsilon \tilde{G}_i - \tilde{\mathbf{v}}_i \cdot \nabla \tilde{\theta}_i - \eta_i \operatorname{div} \tilde{\mathbf{v}}_i - \tilde{\theta}_i \operatorname{div} \tilde{\mathbf{v}}_i, \end{cases} \quad (3.8)$$

where the force \tilde{F}_i and heat source \tilde{G}_i are given by

$$\begin{aligned} \tilde{F}_i &= (\epsilon P_i + \eta_i) \mathbf{f} - (\epsilon P_i + \eta_i)(U_i + \tilde{\mathbf{v}}_i) \cdot \nabla (U_i + \tilde{\mathbf{v}}_i) - \tilde{\theta}_i \nabla P_i - P_i \nabla \tilde{\theta}_i, \\ \tilde{G}_i &= \tilde{\Psi}_i - (\epsilon P_i + \eta_i)(U_i + \tilde{\mathbf{v}}_i) \cdot \nabla \tilde{\theta}_i + (\epsilon P_i + \eta_i) \tilde{\theta}_i \operatorname{div} \tilde{\mathbf{v}}_i + P_i \operatorname{div} \tilde{\mathbf{v}}_i. \end{aligned}$$

Now, if we set

$$\begin{aligned} W &= U_2 - U_1, \quad \tilde{W} = \tilde{U}_2 - \tilde{U}_1, \quad Q = P_2 - P_1, \quad \xi = \eta_2 - \eta_1, \\ \mathbf{w} &= \mathbf{v}_2 - \mathbf{v}_1, \quad \tilde{\mathbf{w}} = \tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1, \quad \beta = \theta_2 - \theta_1, \quad \tilde{\beta} = \tilde{\theta}_2 - \tilde{\theta}_1, \\ J &= \tilde{F}_2 - \tilde{F}_1, \quad I = \tilde{G}_2 - \tilde{G}_1, \end{aligned}$$

then, we can have the following systems:

$$\begin{cases} (\tilde{U}_1 + \tilde{v}_1) \cdot \nabla W - \mu \Delta W + \nabla Q = -(\tilde{W} + \tilde{\mathbf{w}}) \cdot \nabla U_2, & \int Q dx = 0, \\ \operatorname{div} W = 0, \end{cases} \quad (3.9)$$

and

$$\begin{cases} U_1 \cdot \nabla \xi + \frac{\operatorname{div} \mathbf{w}}{\epsilon} = -\operatorname{div}(\tilde{\mathbf{v}}_1 \xi + \tilde{\mathbf{w}} \eta_2) - \epsilon \operatorname{div}(P_1(W + \tilde{\mathbf{w}}) + Q(U_2 + \tilde{\mathbf{v}}_2)) - W \cdot \nabla \eta_2, \\ U_1 \cdot \nabla \mathbf{w} - \mu \Delta \mathbf{w} - \zeta \nabla \operatorname{div} \mathbf{w} + \frac{\nabla \xi + \nabla \beta}{\epsilon} \\ \quad = \epsilon J - \tilde{\mathbf{w}} \cdot \nabla \tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\mathbf{w}} - W \cdot \nabla \mathbf{v}_2 - \nabla(\tilde{\theta}_1 \xi + \tilde{\beta} \eta_2), \\ U_1 \cdot \nabla \beta - \kappa \Delta \beta + \frac{\operatorname{div} \mathbf{w}}{\epsilon} \\ \quad = \epsilon I - \tilde{\mathbf{w}} \cdot \nabla \tilde{\theta}_2 - \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\beta} - W \cdot \nabla \theta_2 - \xi \operatorname{div} \tilde{\mathbf{v}}_1 - \eta_2 \operatorname{div} \tilde{\mathbf{w}} - \tilde{\beta} \operatorname{div} \tilde{\mathbf{v}}_1 - \tilde{\theta}_2 \operatorname{div} \tilde{\mathbf{w}}, \end{cases} \quad (3.10)$$

where J and I read as

$$\begin{aligned} J &= (\epsilon Q + \xi) \mathbf{f} - (\epsilon Q + \xi)(U_2 + \tilde{\mathbf{v}}_2) \cdot \nabla (U_2 + \tilde{\mathbf{v}}_2) \\ &\quad - (\epsilon P_1 + \eta_1)((W + \tilde{\mathbf{w}}) \cdot \nabla (U_2 + \tilde{\mathbf{v}}_2) + (U_1 + \tilde{\mathbf{v}}_1) \cdot \nabla (W + \tilde{\mathbf{w}})) - \nabla(\tilde{\beta} P_1 + Q \tilde{\theta}_2), \\ I &= 2\mu D(W + \tilde{\mathbf{w}}) : D(U_2 + \tilde{\mathbf{v}}_2) + 2\mu D(U_1 + \tilde{\mathbf{v}}_1) : D(W + \tilde{\mathbf{w}}) \\ &\quad + \lambda \operatorname{div}(W + \tilde{\mathbf{w}}) \cdot \operatorname{div}(U_2 + \tilde{\mathbf{v}}_2) + \lambda \operatorname{div}(U_1 + \tilde{\mathbf{v}}_1) \cdot \operatorname{div}(W + \tilde{\mathbf{w}}) \\ &\quad - (\epsilon Q + \xi)(U_2 + \tilde{\mathbf{v}}_2) \cdot \nabla \tilde{\theta}_2 - (\epsilon P_1 + \eta_1)((W + \tilde{\mathbf{w}}) \cdot \nabla \tilde{\theta}_2 + (U_1 + \tilde{\mathbf{v}}_1) \cdot \nabla \tilde{\beta}) \\ &\quad + (\epsilon Q + \xi) \tilde{\theta}_2 \operatorname{div} \tilde{\mathbf{v}}_2 - (\epsilon P_1 + \eta_1)(\tilde{\beta} \operatorname{div} \tilde{\mathbf{v}}_2 + \tilde{\theta}_1 \operatorname{div} \tilde{\mathbf{w}}) + Q \operatorname{div} \tilde{\mathbf{v}}_2 + P_1 \operatorname{div} \tilde{\mathbf{w}}. \end{aligned}$$

Note that J and I can be bounded as follows.

$$\begin{aligned} \|J\|_0 &\leq (\epsilon \|Q\|_1 + \|\xi\|_1)(\|\mathbf{f}\|_2 + \|U_2\|_2^2 + \|\tilde{\mathbf{v}}_2\|_2^2) + (\epsilon \|P_1\|_2 + \|\eta_1\|_2)(\|W\|_1 + \|\tilde{\mathbf{w}}\|_1) \\ &\quad \cdot (\|U_2\|_2 + \|\tilde{\mathbf{v}}_2\|_2 + \|U_1\|_2 + \|\tilde{\mathbf{v}}_1\|_2) + \|\tilde{\theta}_2\|_2 \|Q\|_1 + \|\tilde{\beta}\|_1 \|P_1\|_2 \end{aligned} \quad (3.11)$$

and

$$\begin{aligned}
\|I\|_0 \leq & C\{(\|W\|_1 + \|\tilde{\mathbf{w}}\|_1)(\|U_1\|_2 + \|U_2\|_2 + \|\tilde{\mathbf{v}}_1\|_2 + \|\tilde{\mathbf{v}}_2\|_2) + (\epsilon\|Q\|_1 + \|\xi\|_1)(\|U_2\|_2 + \|\tilde{\mathbf{v}}_2\|_2) \\
& \cdot \|\tilde{\theta}_2\|_2 + (\epsilon\|P_1\|_2 + \|\eta_1\|_2)(\|W\|_1 + \|\tilde{\mathbf{w}}\|_1)\|\tilde{\theta}_2\|_2 + \|U_1\|_2 + \|\tilde{\mathbf{v}}_1\|_2\|\beta\|_1\} \\
& + (\epsilon\|P_1\|_2 + \|\eta_1\|_2)(\|\tilde{\beta}\|_1\|\tilde{\mathbf{v}}_2\|_2 + \|\tilde{\theta}_1\|_2\|\tilde{\mathbf{w}}\|_1) + \|Q\|_1\|\tilde{\mathbf{v}}_2\|_2 + \|P_1\|_2\|\tilde{\mathbf{w}}\|_1\}
\end{aligned} \tag{3.12}$$

On the one hand, we multiply (3.9)₁ by W and make use of Poincaré's inequality to deduce that

$$\left(\mu - \frac{C\|\tilde{\mathbf{v}}\|_3}{2}\right)\|\nabla W\|_0^2 \leq C(\|\tilde{W}\|_1 + \|\tilde{\mathbf{w}}\|_1)\|U_2\|_3\|W\|_1,$$

where C is a positive constant depending only on Ω and μ . Consequently,

$$\|W\|_1 \leq C(\|\tilde{W}\|_1 + \|\tilde{\mathbf{w}}\|_1) \tag{3.13}$$

for some positive constant C depending only on $\Omega, \mu, \lambda, \mathbf{f}, E$ and ϵ_0 .

By the classical estimates for the Stokes equations

$$\begin{cases} -\mu\Delta W + \nabla Q = -(\tilde{W} + \tilde{\mathbf{w}}) \cdot \nabla U_2 - (\tilde{U}_1 + \tilde{\mathbf{v}}_1) \cdot \nabla W, \\ \operatorname{div} W = 0, \end{cases}$$

we obtain that

$$\begin{aligned}
\|W\|_2 + \|\nabla Q\|_0 & \leq C(\|\tilde{W} + \tilde{\mathbf{w}}\|_1\|U_2\|_2 + \|\tilde{U}_1 + \tilde{\mathbf{v}}_1\|_2\|W\|_1) \\
& \leq C(\|\tilde{W}\|_1 + \|\tilde{\mathbf{w}}\|_1),
\end{aligned} \tag{3.14}$$

where the estimate (3.13) has been used.

On the other hand, if we multiply (3.10)₁, (3.10)₂ and (3.10)₃ by ξ , w and β in L^2 respectively, we find that

$$\begin{aligned}
& \mu\|\nabla \mathbf{w}\|_0^2 + \zeta\|\operatorname{div} \mathbf{w}\|_0^2 + \kappa\|\nabla \beta\|_0^2 \\
& = - \int \left[\xi \operatorname{div} \tilde{\mathbf{v}}_1 + \eta_2 \operatorname{div} \tilde{\mathbf{w}} + \tilde{\mathbf{v}}_1 \cdot \nabla \xi + \tilde{\mathbf{w}} \cdot \nabla \eta_2 + \epsilon \operatorname{div}(P_1(W + \tilde{\mathbf{w}}) + Q(U_2 + \tilde{\mathbf{v}}_2)) + W \cdot \nabla \eta_2 \right] \xi dx \\
& \quad + \int \left[\epsilon J - \tilde{\mathbf{w}} \cdot \nabla \tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\mathbf{w}} - W \cdot \nabla \mathbf{v}_2 - \nabla(\tilde{\theta}_1 \xi + \tilde{\beta} \eta_2) \right] \cdot w dx \\
& \quad + \int \left[\epsilon D - \tilde{\mathbf{w}} \cdot \nabla \tilde{\theta}_2 - \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\beta} - W \cdot \nabla \theta_2 - \xi \operatorname{div} \tilde{\mathbf{v}}_1 - \eta_2 \operatorname{div} \tilde{\mathbf{w}} - \tilde{\beta} \operatorname{div} \tilde{\mathbf{v}}_1 - \tilde{\theta}_1 \operatorname{div} \tilde{\mathbf{w}} \right] \beta dx \\
& \leq C \left\{ \|\xi\|_0^2 \|\tilde{\mathbf{v}}_1\|_3 + \|\xi\|_1 \|\tilde{\mathbf{w}}\|_1 \|\eta_2\|_1 + \epsilon \|\xi\|_0 [\|P_1\|_2(\|W\|_1 + \|\tilde{\mathbf{w}}\|_1) + \|Q\|_1(\|U_2\|_2 + \|\tilde{\mathbf{v}}_2\|_2)] \right. \\
& \quad + \|W\|_1 \|\eta_2\|_3 \|\xi\|_1 \left. \right\} + C \left\{ \epsilon \|J\|_0^2 + \|\tilde{\mathbf{w}}\|_1^2 (\|\tilde{\mathbf{v}}_2\|_2^2 + \|\tilde{\mathbf{v}}_1\|_2^2) + \|W\|_1^2 \|\mathbf{v}_2\|_2^2 + \|\tilde{\theta}_2\|_1^2 \|\xi\|_1^2 \right. \\
& \quad + \|\tilde{\beta}\|_1^2 \|\eta\|_2^2 \left. \right\} + C \left\{ \epsilon \|I\|_0^2 + \|\tilde{\mathbf{w}}\|_1^2 \|\tilde{\theta}_2\|_2^2 + \|\tilde{\mathbf{v}}_1\|_2^2 \|\tilde{\beta}\|_1^2 + \|W\|_1^2 \|\theta_2\|_2^2 + \|\xi\|_1^2 \|\tilde{\mathbf{v}}_1\|_2^2 \right. \\
& \quad + \|\eta_2\|_2^2 \|\tilde{\mathbf{w}}\|_1^2 + \|\tilde{\beta}\|_1^2 \|\tilde{\mathbf{v}}_1\|_2^2 + \|\tilde{\theta}_1\|_2^2 \|\tilde{\mathbf{w}}\|_1^2 \left. \right\} + \delta(\|\mathbf{w}\|_0^2 + \|\beta\|_0^2).
\end{aligned} \tag{3.15}$$

Also, from the Stokes equations

$$\begin{cases} -\mu\epsilon\Delta \mathbf{w} + \nabla \xi = \epsilon(\epsilon J - \tilde{\mathbf{w}} \cdot \nabla \tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\mathbf{w}} - W \cdot \nabla \mathbf{v}_2 - \nabla(\tilde{\theta}_1 \xi + \tilde{\beta} \eta_2) + \zeta \nabla \operatorname{div} \mathbf{w}) - \nabla \beta, \\ \operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{w}, \end{cases}$$

we get the following estimate

$$\begin{aligned}
\epsilon\|\mathbf{w}\|_2 + \|\nabla \xi\|_0 & \leq \epsilon C \left(\|\operatorname{div} \mathbf{w}\|_2 + \epsilon \|J\|_0 + \|\tilde{\mathbf{w}}\|_1(\|\tilde{\mathbf{v}}_2\|_2 + \|\tilde{\mathbf{v}}_1\|_2) + \|W\|_1 \|\mathbf{v}_2\|_2 \right. \\
& \quad \left. + \|\tilde{\theta}_1\|_2 \|\xi\|_1 + \|\tilde{\beta}\|_1 \|\eta_2\|_2 \right) + C\|\beta\|_1.
\end{aligned} \tag{3.16}$$

Applying Poincaré's inequality and substituting (3.16) into (3.15), employing the estimates (3.11)–(3.14) and recalling the smallness of ϵ_0 and E , we conclude that

$$\|W\|_1 + \|\mathbf{w}\|_1 + \|\beta\|_1 \leq C(\|\tilde{W}\|_1 + \|\tilde{\mathbf{w}}\|_1 + \|\tilde{\beta}\|_1),$$

where C is a positive constant depending only on $\Omega, \mu, \lambda, \mathbf{f}, E$ and ϵ_0 . This completes the proof. \square

Finally, having had Lemmas 3.1 and 3.2, we can apply the Tikhonov fixed point theorem to find a fixed point $(U, \mathbf{v}, \theta) = N(U, \mathbf{v}, \theta)$ in the set K . Moreover, the pressure $P \in \bar{H}^2$ satisfies

$$\nabla P = \mathbf{f} + \mathbf{g} + \mu \Delta U - (U + \mathbf{v}) \cdot \nabla U,$$

and $(U + \mathbf{v}, \epsilon P + \eta, \theta)$ is a solution to (1.5). Thus, we have shown the following proposition.

Proposition 3.1. *Let $\mathbf{f}, \mathbf{g} \in H^2(\Omega)$. Then, there exists an ϵ_0 depending only on $\Omega, \mu, \lambda, \mathbf{f}$ and \mathbf{g} , such that for all $\epsilon \in (0, \epsilon_0)$, there is a solution $(U, P, \mathbf{v}, \eta, \theta) \in (H^4 \cap H_{0,\sigma}^1) \times \bar{H}^3 \times (H^3 \cap H_0^1) \times \bar{H}^2 \times (H^3 \cap H_0^1)$ of (2.1) and (2.2), satisfying*

$$\|\mathbf{v}\|_3 + \|\eta\|_2 + \|\theta\|_3 \leq E,$$

where E is a small positive constant depending only on $\Omega, \mu, \lambda, \mathbf{f}$ and \mathbf{g} . Moreover, $(U + \mathbf{v}, \epsilon P + \eta, \theta)$ is a solution of the system (1.5) for any $\epsilon \in (0, \epsilon_0)$.

4 Incompressible limit

Let $\epsilon < \epsilon_0$ and $(U^\epsilon, \mathbf{v}^\epsilon, \theta^\epsilon) \in K$ be the solution established in Proposition 3.1. We take $\mathbf{v} = \tilde{\mathbf{v}} = \mathbf{v}^\epsilon$, $\theta = \tilde{\theta} = \theta^\epsilon$ and $\eta = \eta^\epsilon$ in (3.6) to get that

$$\begin{aligned} \|\mathbf{v}^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 &\leq C_{16}(1 + M_1)^5 \left\{ \epsilon \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) + \delta \right\} \|\eta\|_2 \\ &\quad + (\epsilon M_1^{\frac{1}{2}} + E) \|\mathbf{v}\|_3 + (\delta M_1^{\frac{1}{2}} + E) \|\theta\|_3 \\ &\quad + C_{16}(1 + M_1)^5 \left\{ \epsilon (M_1 + 1)^2 + \epsilon (1 + M_1) \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] \right\}. \end{aligned}$$

Thus, by taking ϵ_0 and E so small that

$$C_{16}(1 + M_1)^5 \left\{ \epsilon \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) \right\} < 1, \quad \epsilon M_1^{\frac{1}{2}} + E < 1,$$

we obtain

$$\|\mathbf{v}^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 \leq C_{16}(1 + M_1)^5 \left\{ \epsilon (M_1 + 1)^2 + \epsilon (1 + M_1) \left[\|\mathbf{f}\|_1 + (M_1 + 1)^2 \right] \right\},$$

whence,

$$\|\mathbf{v}^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (4.1)$$

Furthermore, from (2.2)₁, i.e.,

$$U^\epsilon \cdot \nabla \eta^\epsilon + \frac{\operatorname{div} \mathbf{v}^\epsilon}{\epsilon} = -\mathbf{v}^\epsilon \cdot \nabla \eta^\epsilon - \eta^\epsilon \operatorname{div} \mathbf{v}^\epsilon - \epsilon \operatorname{div} (P^\epsilon (U^\epsilon + \mathbf{v}^\epsilon))$$

and (4.1) we get that as $\epsilon \rightarrow 0$,

$$\left\| \frac{\operatorname{div} \mathbf{v}^\epsilon}{\epsilon} \right\|_1 \leq \|\mathbf{v}^\epsilon \cdot \nabla \eta^\epsilon\|_1 + \|\eta^\epsilon \operatorname{div} \mathbf{v}^\epsilon\|_1 + \|\epsilon \operatorname{div} (P^\epsilon (U^\epsilon + \mathbf{v}^\epsilon))\|_1 + \|U^\epsilon \cdot \nabla \eta^\epsilon\|_1 \rightarrow 0. \quad (4.2)$$

Due to (4.1) and

$$\frac{\nabla\eta^\epsilon + \nabla\theta^\epsilon}{\epsilon} = \epsilon F^\epsilon - \mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon - \theta^\epsilon \nabla \eta^\epsilon - \eta^\epsilon \nabla \theta^\epsilon - U^\epsilon \cdot \nabla \mathbf{v}^\epsilon + \mu \Delta \mathbf{v}^\epsilon + (\mu + \lambda) \nabla \operatorname{div} \mathbf{v}^\epsilon$$

with $F^\epsilon = (\epsilon P^\epsilon + \eta^\epsilon) \mathbf{f} - (\epsilon P^\epsilon + \eta^\epsilon)(U^\epsilon + \mathbf{v}^\epsilon) \cdot \nabla (U^\epsilon + \mathbf{v}^\epsilon) - \theta^\epsilon \nabla P^\epsilon - P^\epsilon \nabla \theta^\epsilon$, which comes from the transform of (2.2)₂, one deduces, recalling Poincaré's inequality, that

$$\left\| \frac{\eta^\epsilon + \theta^\epsilon}{\epsilon} \right\|_2 \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (4.3)$$

On the other hand, in view of Lemma 2.2, we observe that (U^ϵ, P^ϵ) is a uniform-in- ϵ bounded sequence in $(H^4 \cap H_0^1) \times \bar{H}^3$. Hence, there are a subsequence of $(U^{\epsilon_k}, P^{\epsilon_k})$, still denoted by $(U^{\epsilon_k}, P^{\epsilon_k})$ for simplicity, and $(\bar{U}, \bar{P}) \in (H^4 \cap H_0^1) \times \bar{H}^3$, such that as $\epsilon \rightarrow 0$,

$$(U^\epsilon, P^\epsilon) \rightharpoonup (\bar{U}, \bar{P}), \quad \text{weakly in } (H^4 \cap H_0^1) \times \bar{H}^3,$$

and

$$(U^\epsilon, P^\epsilon) \rightarrow (\bar{U}, \bar{P}), \quad \text{strongly in } (H^3 \cap H_0^1) \times \bar{H}^2.$$

Thus, if we take to the limit as $\epsilon \rightarrow 0$ in (2.1) and (2.2), we conclude that (\bar{U}, \bar{P}) is a solution of the steady incompressible Navier-Stokes equations (1.7).

In conclusion, we have that

$$\lim_{\epsilon \rightarrow 0} \inf_{U, P \in \mathbf{L}} \|U^\epsilon + \mathbf{v}^\epsilon - U\|_3 + \left\| P^\epsilon + \frac{\eta^\epsilon + \theta^\epsilon}{\epsilon} - P \right\|_2 + \|\theta^\epsilon\|_3 = 0,$$

where \mathbf{L} is the same as in Theorem 1.1. Thus, the proof of the low Mach number limit is completed.

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